

Neighbourhood counting and new bounds for small three-colour Ramsey numbers

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Abstract

We improve four of the upper bounds for small three-colour Ramsey numbers listed in Radziszowski's survey: $R(4, 4, 4)$ from 229 to 198, $R(3, 4, 4)$ from 77 to 73, $R(3, 4, 5)$ from 157 to 137 and $R(3, 3, 6)$ from 91 to 86. Each upper bound $R(t_1, t_2, t_3) \leq n$ is proved by contradiction from a colouring of the edges of K_n with, for each i , no clique of order t_i in colour i . Ramsey bounds inside colour neighbourhoods restrict the degree profile of each vertex, its triple of colour degrees, to a finite list of admissible profiles. A counting identity, or a flag-algebra certificate whose constraints include these local bounds, then shows that an explicit function of the profile, positive on every admissible profile, has nonpositive average, a contradiction. The four improvements come from five-vertex certificates, checked in exact arithmetic by computer.

Some bounds also have proofs by hand: a double count of about a page gives $R(3, 3, 6) \leq 88$, and weighted triangle and four-vertex identities give $R(3, 4, 5) \leq 155$ and $R(4, 4, 4) \leq 226$.

We also give upper bounds for numbers for which the survey lists none: $R(3, 3, k)$ for $7 \leq k \leq 15$, for instance $R(3, 3, 8) \leq 173$, as well as $R(3, 4, k)$ for $6 \leq k \leq 8$, $R(3, 5, 5)$ and $R(3, 5, 6)$ (Table 1). Through the four-colour recurrence, our bounds for $R(3, 4, 4)$ and $R(4, 4, 4)$, with the survey's $R(3, 3, 3, 4) \leq 149$, also give $R(3, 3, 4, 4) \leq 442$, $R(3, 4, 4, 4) \leq 1521$ and $R(4, 4, 4, 4) \leq 6082$, improving 450, 1576 and 6301. Explicit colourings give $R(3, 3, k) \geq 133, 183, 203, 232, 267$ and 278 for $k = 9, 11, 12, 13, 14$ and 15 , improving the lower bounds in the survey.

1 Introduction

For integers $t_1, t_2, t_3 \geq 1$, the Ramsey number $R(t_1, t_2, t_3)$ is the least n such that every colouring of the edges of K_n with colours 1, 2, 3 contains, for some i , a K_{t_i} all of whose edges have colour i . Two-colour numbers $R(s, t)$ are defined in the same way. We call colours 1, 2 and 3 red, blue and green, respectively.

Renaming the colours shows that $R(t_1, t_2, t_3)$ is symmetric in its arguments. Up to this symmetry, only two values with every $t_i \geq 3$ are known exactly: $R(3, 3, 3) = 17$ [13] and $R(3, 3, 4) = 30$ [6]. See Radziszowski's dynamic survey [20] for the current state of knowledge.

The neighbourhood recurrence. Most upper bounds for three-colour numbers come from the neighbourhood recurrence, implicit in [13],

$$R(\mathbf{t}) \leq R(\mathbf{t} - \mathbf{e}_1) + R(\mathbf{t} - \mathbf{e}_2) + R(\mathbf{t} - \mathbf{e}_3) - 1, \quad (1)$$

where $\mathbf{t} = (t_1, t_2, t_3)$ and $\mathbf{e}_i$ is the i th unit vector.

Write $N_i(v)$ for the colour- i neighbourhood of a vertex v , $d_i(v) = |N_i(v)|$, and $\mathbf{d}(v) = (d_1(v), d_2(v), d_3(v))$ for its *degree profile*.

In a colouring of K_n with no colour- i K_{t_i} for any i , the set $N_i(v)$ has no colour- i K_{t_i-1} , which would extend by v , and no colour- j K_{t_j} for $j \neq i$. So

$$d_i(v) \leq R(\mathbf{t} - \mathbf{e}_i) - 1,$$

and summing over i gives

$$n - 1 \leq \sum_i (R(\mathbf{t} - \mathbf{e}_i) - 1),$$

that is, n is less than the right-hand side of (1).

An entry equal to 2 can be dropped, leaving a two-colour Ramsey number, as a colouring with no colour- i K_2 has no colour- i edges. For instance (1) gives

$$R(3, 3, k) \leq 2R(3, k) + R(3, 3, k - 1) - 1.$$

Parity and other improvements. The recurrence has a *parity refinement*: it is strict when its right-hand side is even and at least one term $R(\mathbf{t} - \mathbf{e}_i)$ is even [20, 6.1(a)]. Indeed, at the order n one less than the right-hand side, $n - 1 = \sum_i (R(\mathbf{t} - \mathbf{e}_i) - 1)$, so equality holds in every cap $d_i(v) \leq R(\mathbf{t} - \mathbf{e}_i) - 1$. For an i with $R(\mathbf{t} - \mathbf{e}_i)$ even, every vertex then has odd colour- i degree, which contradicts the handshake lemma as n is odd.

Both statements, and every cap below, hold with the same proofs when a Ramsey number is replaced by a proven upper bound for it. For example, $R(3, 3, 6) \leq 91$ [20, 5] follows from $R(3, 6) = 18$ and $R(3, 3, 5) \leq 57$ (itself (1) with $R(3, 5) = 14$ and $R(3, 3, 4) = 30$). Here the right-hand side is $18 + 18 + 57 - 1 = 92$, and 92 and 18 are even.

Only a few three-colour upper bounds improve on (1) with parity [20]. Among them are Boza's recent $R(3, 4, 5) \leq 157$ and $R(4, 4, 4) \leq 229$ [5], obtained by an analogue of the parity argument modulo 3. From $R(4, 5) = 25$, $R(3, 3, 5) \leq 57$ and $R(3, 4, 4) \leq 77$, the recurrence gives $25 + 57 + 77 - 1 = 158$ and $3 \cdot 77 - 1 = 230$ for these two numbers. Parity does not apply, since every term is odd. The survey also lists $R(3, 3, 6) \leq 91$, which [5] derives in the same way, among the bounds that improve on the recurrence with parity. But, as shown above, it already follows from (1) with parity. The bound $R(3, 4, 4) \leq 77$ is itself (1), $18 + 30 + 30 - 1$, and has not been improved before.

Our approach. The recurrence bounds the colour neighbourhoods of one vertex at a time. The parity refinement looks at all vertices together, but only at the order one less than the right-hand side of (1), where every vertex has the *maximal profile*

$$(R(\mathbf{t} - \mathbf{e}_1) - 1, R(\mathbf{t} - \mathbf{e}_2) - 1, R(\mathbf{t} - \mathbf{e}_3) - 1),$$

and only at the parity of its entries. Our arguments also look at all vertices together, but at smaller orders, where several profiles are possible, and they count small substructures instead.

1.1 Results

Our upper bounds are in Table 1, and our lower bounds, with those listed in the survey [20], in Table 2.

Besides the exact values of $R(3, 3, 3)$ and $R(3, 3, 4)$, the survey lists upper bounds for only five three-colour numbers with every $t_i \geq 3$: $R(3, 3, 5)$, $R(3, 3, 6)$, $R(3, 4, 4)$, $R(3, 4, 5)$ and $R(4, 4, 4)$. We improve four of them, the bounds $R(3, 3, 6) \leq 91$, $R(3, 4, 5) \leq 157$ and $R(4, 4, 4) \leq 229$ recalled above and $R(3, 4, 4) \leq 77$. We do not improve $R(3, 3, 5) \leq 57$.

Benchmarks. For the other three-colour numbers in Table 1 no upper bound is listed. So the table gives instead a *benchmark*, marked with a dagger: the result of applying (1) with parity repeatedly, starting from the values listed in [20, 5] and never using our new bounds. For instance, $R(3, 7) = 23$ gives $23 + 23 + 91 - 1 = 136$ for $R(3, 3, 7)$, and then $R(3, 8) = 28$ gives $28 + 28 + 136 - 1 = 191$ for $R(3, 3, 8)$. Parity improves neither. We do not claim that any benchmark has appeared in the literature.

In the main part of the table these starting values, and all inputs of our bounds, are published. The survey also reports four two-colour bounds as unpublished, marked * in Table 3. They lower four benchmarks and some of our bounds, and the lower part of the table gives these variants.

The benchmarks use the bounds of [5], but not the criterion modulo 3 that proves them [5, Theorem 2.1]. Stated for exact values, it lowers the right-hand side P of (1) by one when three conditions hold: $P \not\equiv 1 \pmod{3}$; for some i with $t_i \geq 4$, $R(\mathbf{t} - \mathbf{e}_i)$ equals its own right-hand side and is $\not\equiv 1 \pmod{3}$; and, for the same i , $R(\mathbf{t} - 2\mathbf{e}_i) \not\equiv 1 \pmod{3}$. Applied in the same way to the benchmarks, it lowers $2 \cdot 50 + 342 - 1 = 441$ for $R(3, 3, 11)$ to 440 (one may assume that the bounds involved are attained, as otherwise (1) already gives 440). Here $342 = 2 \cdot 41 + 261 - 1$ is exactly the right-hand side of (1) for $R(3, 3, 10)$, which parity does not lower, and 441, 342 and 261 are all divisible by 3, so none is $\equiv 1 \pmod{3}$. From 440, (1) with parity gives 557, 691, 844 and 1017 for $12 \leq k \leq 15$, in place of 558, 693, 846 and 1019. No other benchmark changes.

Hand and computer bounds. The bounds marked “hand” come from Sections 4, 6 and 7: double counting, which a reader can check with pen and paper. Those marked “computer” come from Section 8:

- five-vertex certificates;
- for $R(3, 3, k)$ with $k \geq 8$, the procedure of Section 5 fed with exclusions and edge-count lower bounds from four- and five-vertex certificates; and
- for four colours, the four-colour recurrence applied to two of these bounds (Corollary 8.7).

The certificates are too large to check by hand but are verified in exact rational arithmetic by programs that share no code with the search that found them. For $9 \leq k \leq 15$, Table 5 gives the bounds on $R(3, 3, k)$ both without the certificates (Theorem 5.2, which is computer-assisted) and with them. Theorem 7.5 also gives $R(4, 4, 4) \leq 225$, from a four-vertex certificate with a single square.

Four colours. With the four-colour form of (1), the bounds $R(3, 4, 4) \leq 73$ and $R(4, 4, 4) \leq 198$, with the survey’s $R(3, 3, 3, 4) \leq 149$, give $R(3, 3, 4, 4) \leq 442$, $R(3, 4, 4, 4) \leq 1521$ and $R(4, 4, 4, 4) \leq 6082$ (Corollary 8.7). The best previous bounds, recalled in Section 8.3, are 450, 1576 and 6301.

number	comparison	hand	computer	where
R(3, 3, 6)	91 [20, 5]	88	86	Theorems 4.1, 8.4
R(3, 3, 7)	136 [†]	129	125	Theorems 4.3, 8.4
R(3, 3, 8)	191 [†]	179	173	Theorems 4.5, 8.5
R(3, 3, k), $9 \leq k \leq 15$	261, ..., 1019 [†]		Table 5	Theorems 5.2, 8.5
R(3, 4, 4)	77 [20]		73	Theorem 8.3
R(3, 4, 5)	157 [5]	155	137	Theorems 6.2, 8.3
R(3, 4, 6)	288 [†]	276	247	Theorems 6.3, 8.3
R(3, 5, 5)	359 [†]	346	316	Theorems 6.4, 8.3
R(3, 4, 7)	483 [†]	458	409	Proposition 6.5, Theorem 8.3
R(3, 4, 8)	757 [†]	705	626	Proposition 6.5, Theorem 8.3
R(3, 5, 6)	733 [†]	685	622	Proposition 6.5, Theorem 8.3
R(4, 4, 4)	229 [5]	226	198	Theorems 7.4, 8.2
R(3, 3, 4, 4)	450 [20]		442	Corollary 8.7
R(3, 4, 4, 4)	1576 [20, 5]		1521	Corollary 8.7
R(4, 4, 4, 4)	6301 [20, 5]		6082	Corollary 8.7

With the unpublished two-colour bounds marked * in Table 3:

number	comparison	hand	computer	where
R(3, 4, 6)	287 [†]	276	245	Theorems 6.3, 8.3
R(3, 4, 7)	479 [†]	456	406	Proposition 6.5, Theorem 8.3
R(3, 4, 8)	748 [†]	700	625	Proposition 6.5, Theorem 8.3
R(3, 5, 6)	730 [†]	683	619	Proposition 6.5, Theorem 8.3

Table 1: Upper bounds. [†] Benchmark where [20] lists no bound: (1) with parity, iterated from the values in [20, 5] (for $9 \leq k \leq 15$, the row “recurrence from 91” of Table 4). The main part of the table uses only published inputs, and the lower part gives the variants with the two-colour bounds that the survey reports as unpublished.

1.2 Method

Each of our three-colour upper bounds $R(\mathbf{t}) \leq n$ is proved by contradiction from a colouring of K_n with no colour- i K_{t_i} for any i , in three steps.

- (i) *Local caps.* By the proof of (1), every profile is *admissible*, that is, a triple of nonnegative integers with sum $n - 1$ and i th entry at most $R(\mathbf{t} - \mathbf{e}_i) - 1$. If n is δ less than the right-hand side of (1), the shortfalls $R(\mathbf{t} - \mathbf{e}_i) - 1 - d_i(v)$, which are nonnegative, therefore sum to $\delta - 1$. So for small δ few profiles are admissible. We also use *mixed* caps on intersections of neighbourhoods of several vertices (Lemmas 2.1 and 2.3), such as

$$|N_i(v) \cap N_j(u)| \leq R(\mathbf{t} - \mathbf{e}_i - \mathbf{e}_j) - 1 \quad \text{for } u \in N_i(v).$$

- (ii) *A counting identity, or a certificate.* A small substructure is counted in two ways: triangles, two-edge paths or triples in Sections 3–5; the nine triangle types that remain when red is triangle-free, in Section 6; four-vertex patterns, as probabilities for a random ordered quadruple of vertices, in Section 7. One side depends only on the degree profiles. The other involves terms such as the number of colour- j edges inside $N_i(v)$, which the mixed caps bound from above (Corollary 2.2). We merge two colours, or weight the

number	lower bound in [20]	new lower bound	where
$R(3, 3, 9)$	129	133	Theorem 9.1
$R(3, 3, 11)$	174	183	Theorem 9.1
$R(3, 3, 12)$	194	203	Theorem 9.1
$R(3, 3, 13)$	217	232	Theorem 9.1
$R(3, 3, 14)$	242	267	Theorem 9.1
$R(3, 3, 15)$	269	278	Theorem 9.3

Table 2: Lower bounds for $R(3, 3, k)$ from explicit colourings. For $k \leq 14$, a known circulant colouring of K_n showing $R(5, k) \geq n + 1$ has its K_5 -free colour split into two triangle-free colours (Section 9). For $k = 15$ eleven vertices are added to another split of the circulant colouring used for $k = 14$ (Section 9.1). We have no new lower bound for other $R(3, 3, k)$.

triangle types, so that every term not determined by the profiles cancels or can be bounded in the required direction. In Section 8 the identity is replaced by a flag-algebra certificate [21], mostly on five vertices: a nonnegative combination of caps, lower bounds on colour degrees and squares that leaves only terms determined by the profiles. Either way we obtain

$$\sum_v \Phi(\mathbf{d}(v)) \leq 0$$

for an explicit function Φ of the profile.

(iii) *A finite check.* Φ is positive at every admissible profile, a contradiction.

Relation to earlier work. The prototype is Goodman’s count of monochromatic triangles in two-colourings [10], based on the identity that the number of non-monochromatic triangles is $\frac{1}{2} \sum_v d_1(v)d_2(v)$. See [11] for three colours.

Bounding the triangles in the union of the triangle-free colours through common neighbourhoods is a standard step in asymptotic work on $R(3, \dots, 3, k)$ [1, 14, 2]. Recently, [2] determined the order of magnitude of $R(3, 3, k)$. What is new here is turning such counts into explicit bounds for small numbers.

The flag-algebra tables of Lidický and Pfender [19], which use no local caps, give only 85 for $R(3, 4, 4)$ and 406 for $R(3, 4, 5)$, above the bounds from the recurrence. Here the caps make the difference.

Using earlier bounds. The bounds build on each other: a new bound caps colour degrees in later proofs. In Sections 4 and 5, for $R(3, 3, k)$ with $k \geq 7$, the argument for $k - 1$ enters the proof for k twice:

- Its bound caps $d_3(v)$, for instance $d_3(v) \leq 87$ in the proof of $R(3, 3, 7) \leq 129$.
- Applied to the colouring on $N_3(v)$, which has no red or blue triangle and no green K_{k-1} , it bounds from below the number of red or blue edges inside $N_3(v)$ (Lemma 3.4(a)). The density inequality (Lemma 3.3) counts vertex triples: six times the number of triangles whose edges are all red or blue equals a sum of terms determined by the profiles plus twice the sum over all v of the number of red or blue edges inside $N_3(v)$. The mixed caps bound those triangles from above, so these edge counts need lower bounds.

The proof of Theorem 4.1 needs no bound of this kind.

1.3 What is proved by hand and what by computer

Hand proofs. The bounds marked “hand” in Table 1 have complete proofs that can be checked by hand, once their inputs are granted. Some inputs, such as $R(3, 3, 4) = 30$ [6], were themselves proved by computer. Besides earlier results of this paper, these inputs are:

- the unstarred entries of Table 3 (Section 2);
- the published two-colour bounds listed in its caption;
- for the first three bounds of Proposition 6.5 only, the starred entries.

Computer-assisted proofs. The other upper bounds are computer-assisted: Theorems 5.2 and 7.5, and the results of Section 8. Among the latter, Corollary 8.7 applies the four-colour recurrence by hand to two of them. Floating-point solvers were used only to find certificates and in the negative experiments we report. Every certificate is verified in exact rational arithmetic. Theorem 7.5 uses the computer only for an exact check of one linear identity among the probabilities of colour patterns on a random ordered quadruple of vertices.

Lower bounds. The lower bounds rest on explicit witnesses, checked by exact searches. For $R(3, 3, 15)$ the green cliques are also bounded through a certifier argument (Lemma 9.2, Section 9.1), which needs no clique search in the extended colouring.

No upper-bound argument of ours, inputs aside, uses a SAT unsatisfiability result or a catalogue of critical colourings. The checking programs are in the ancillary files, and Section 10 lists the few that use libraries beyond the standard ones.

How to read this paper. The strongest upper bounds come from Section 8. Theorems 8.2 and 8.4, the bounds $R(3, 4, 4) \leq 73$ and $R(3, 4, 5) \leq 137$ of Theorem 8.3, and Corollary 8.7 need only Section 2, the set-up of Section 7 (the random quadruple, Lemma 7.1 and the square Q) and Section 8. These results include the four improvements on three-colour bounds listed in the survey, and the four-colour bounds of Corollary 8.7. Some of the other bounds of Theorem 8.3 also take caps from the hand bounds of Section 4, and Theorem 8.5 also uses the lemmas of Section 3 and the procedure of Section 5.

- Sections 3, 4 and 6, and Theorem 7.4, give independent proofs by hand of weaker bounds, which a reader can check with pen and paper.
- Section 5 iterates the arguments of Sections 3 and 4 by computer, for $9 \leq k \leq 15$.
- Section 9, on lower bounds, is independent of the rest.

2 Preliminaries

Notation. Let $\mathbf{t} = (t_1, t_2, t_3)$. A colouring of $E(K_n)$ is $\mathbf{t}$ -*avoiding*, or a $(\mathbf{t}; n)$ -*colouring* when we record the order, if for no i is there a K_{t_i} in colour i . Restriction to fewer vertices preserves this property. So a $(\mathbf{t}; n)$ -colouring exists if and only if $n < R(\mathbf{t})$, and a vertex set on which a colouring is $\mathbf{t}$ -avoiding has at most $R(\mathbf{t}) - 1$ vertices.

Besides $R(2, a, b) = R(a, b)$ from the introduction, so that $R(2, 2, a) = a$, we use $R(1, a, b) = 1$, as a single vertex is vacuously a K_1 in colour 1. Such degenerate values arise in Lemmas 2.1 and 2.3 below when an entry drops to 2 or 1.

For a vertex set S , $e_j(S)$ is the number of colour- j edges inside S . It is unrelated to the unit vectors $\mathbf{e}_i$.

Proven bounds in place of Ramsey numbers. Our upper-bound arguments use a Ramsey number $R(\mathbf{s})$ only through the cap

$$|S| \leq R(\mathbf{s}) - 1$$

for a set S on which the colouring is $\mathbf{s}$ -avoiding. So a proven bound $R(\mathbf{s}) \leq B$ may replace $R(\mathbf{s})$ there, as in $R(3, 3, 5) - 1 \leq 56$. Unless a proof says otherwise, B is

- an unstarred entry of Table 3,
- one of the published bounds listed in its caption, or
- a bound proved earlier in this paper.

The symbol $R(\cdot)$ itself always denotes the true value.

Lemma 2.1 (Local caps). *Let $t_1, t_2, t_3 \geq 3$, let c be a $\mathbf{t}$ -avoiding colouring, v a vertex and i a colour. Then $d_i(v) \leq D_i := R(\mathbf{t} - \mathbf{e}_i) - 1$, and for every colour j and every $u \in N_i(v)$,*

$$|N_i(v) \cap N_j(u)| \leq C_{ij} := R(\mathbf{t} - \mathbf{e}_i - \mathbf{e}_j) - 1.$$

For instance, for $\mathbf{t} = (3, 3, k)$,

$$\begin{aligned} D_1 &= D_2 = R(3, k) - 1, \\ C_{12} &= C_{21} = R(2, 2, k) - 1 = k - 1 \quad \text{and} \quad C_{11} = C_{22} = R(1, 3, k) - 1 = 0, \end{aligned}$$

the last restating that red and blue are triangle-free.

Where a number is given below for a cap D_i or C_{ij} whose Ramsey number is unknown, such as $D_3 = 56$ for $\mathbf{t} = (3, 3, 6)$, it is the value $B - 1$ from a bound $R(\mathbf{s}) \leq B$ as above. The lemma holds with it since $R(\mathbf{s}) \leq B$.

Proof. Let $\mathbf{s} \leq \mathbf{t}$ entrywise, and let S be a vertex set that contains, for each l with $s_l < t_l$, no colour- l K_{s_l} . As c is $\mathbf{t}$ -avoiding, its restriction to S is then $\mathbf{s}$ -avoiding, and so $|S| \leq R(\mathbf{s}) - 1$.

For $S = N_i(v)$ and $\mathbf{s} = \mathbf{t} - \mathbf{e}_i$, a colour- i K_{t_i-1} in S would extend by v , so $d_i(v) \leq D_i$.

For $S = N_i(v) \cap N_j(u)$ and $\mathbf{s} = \mathbf{t} - \mathbf{e}_i - \mathbf{e}_j$, whose entries are positive as every $t_l \geq 3$:

- if $i \neq j$, a colour- i K_{t_i-1} in S would extend by v , and a colour- j K_{t_j-1} by u ;
- if $i = j$, then uv has colour i , so a colour- i K_{t_i-2} in S would extend by both u and v .

So $|S| \leq C_{ij}$. □

Corollary 2.2. *With c , v and i as in Lemma 2.1 and j any colour,*

$$e_j(N_i(v)) \leq \lfloor C_{ij} d_i(v) / 2 \rfloor.$$

Proof. Each $u \in N_i(v)$ has $|N_i(v) \cap N_j(u)| \leq C_{ij}$ colour- j neighbours in $N_i(v)$. Apply the handshake lemma. $\square$

The certificates of Section 8 use caps on intersections of more than two neighbourhoods. They come from the same argument.

Lemma 2.3 (Caps on several neighbourhoods). *Let c be a $\mathbf{t}$ -avoiding colouring, Y a nonempty set of vertices and a a map from Y to the colours. For each colour i , let w_i be the largest order of a colour- i clique contained in $Y_i = \{y \in Y : a(y) = i\}$ (so $w_i = 0$ if Y_i is empty), and let $\mathbf{w} = (w_1, w_2, w_3)$. Then $S = \bigcap_{y \in Y} N_{a(y)}(y)$ satisfies*

$$|S| \leq R(\mathbf{t} - \mathbf{w}) - 1.$$

Proof. As c is $\mathbf{t}$ -avoiding, $w_i \leq t_i - 1$, so every entry of $\mathbf{t} - \mathbf{w}$ is at least 1. Every vertex of S is joined in colour i to every vertex of Y_i , and S is disjoint from Y , as $y \notin N_{a(y)}(y)$. So a colour- i clique in S , together with a colour- i clique of order w_i in Y_i , is a colour- i clique. Hence S contains no colour- i $K_{t_i - w_i}$, the colouring on S is $(\mathbf{t} - \mathbf{w})$ -avoiding, and $|S| \leq R(\mathbf{t} - \mathbf{w}) - 1$. $\square$

Lemma 2.1 is the case $|Y| \leq 2$:

- $Y = \{v\}$ with $a(v) = i$ gives $\mathbf{w} = \mathbf{e}_i$ and the cap D_i ; and
- for $Y = \{v, u\}$ with $u \in N_i(v)$, $a(v) = i$ and $a(u) = j$, the vector $\mathbf{w}$ is $\mathbf{e}_i + \mathbf{e}_j$ if $j \neq i$, and $2\mathbf{e}_i$ if $j = i$, as uv then has colour i .

When $w_i = t_i - 1$ for some i , the cap is $R(\mathbf{t} - \mathbf{w}) - 1 = 0$.

Triangle types and corners. In a 3-colouring of K_n , write

- M_i for the number of triangles with all edges of colour i ,
- $T_{i,j}$ ($i \neq j$) for the number with two edges of colour i , which meet at the *apex* of the triangle, and one of colour j , and
- X for the number of rainbow triangles, whose edges have three different colours.

These count the ten *types* of triangle.

A *corner* at v is a pair of edges vu, vw , with third edge uw . It is an $\{i, j\}$ -*corner* if vu and vw have colours i and j in some order. The number of $\{i, i\}$ -corners at v is $\binom{d_i(v)}{2}$, and that of $\{i, j\}$ -corners with $j \neq i$ is $d_i(v)d_j(v)$. For every colour j , including $j = i$, the $\{i, i\}$ -corners at v whose third edge has colour j correspond to the colour- j edges inside $N_i(v)$, so there are $e_j(N_i(v))$ of them.

On the other hand,

- a triangle of colour i contains three $\{i, i\}$ -corners;
- one of type $T_{i,j}$ contains one $\{i, i\}$ -corner, at its apex, whose third edge has colour j , and two $\{i, j\}$ -corners; and

- a rainbow triangle contains one $\{i, j\}$ -corner for each of the three pairs $\{i, j\}$ with $i \neq j$.

Counting corners at the vertices and in the triangles gives the following identities.

Lemma 2.4 (Corner identities). *In every 3-colouring of K_n , for every colour i ,*

$$\sum_v \binom{d_i(v)}{2} = 3M_i + \sum_{l \neq i} T_{i,l}, \quad 3M_i = \sum_v e_i(N_i(v)),$$

and for every colour $j \neq i$,

$$\sum_v d_i(v)d_j(v) = 2T_{i,j} + 2T_{j,i} + X, \quad T_{i,j} = \sum_v e_j(N_i(v)).$$

two colours	$R(3, 3) = 6, R(3, 4) = 9, R(3, 5) = 14, R(3, 6) = 18, R(3, 7) = 23, R(3, 8) = 28,$ $R(3, 9) = 36, R(3, 10) \leq 41$ [3], $R(3, 11) \leq 50$ [9], $R(3, 12) \leq 59$ [18], $R(3, 13) \leq$ $68, R(3, 14) \leq 77, R(3, 15) \leq 87$ [9]; $R(4, 4) = 18, R(4, 5) = 25, R(5, 5) \leq 46$ [4]; and $R(4, 6) \leq 40^*, R(4, 7) \leq 58^*, R(4, 8) \leq 79^*, R(5, 6) \leq 85^*$
three colours	$R(3, 3, 3) = 17$ [13], $R(3, 3, 4) = 30$ [6]; $R(3, 3, 5) \leq 57$ and $R(3, 4, 4) \leq 77$, both from (1)

Table 3: Inputs, as listed in [20]. The values $R(3, k)$ are given for $3 \leq k \leq 15$ since $R(3, k) - 1$ is the cap $D_1 = D_2$ for $\mathbf{t} = (3, 3, k)$, and the procedure of Section 5 runs through each of these k .

* Reported in [20, Table Ib] as unpublished (Angeltveit–McKay, personal communication); the published bounds are $R(4, 6) \leq 41, R(4, 7) \leq 61, R(4, 8) \leq 84$ and $R(5, 6) \leq 87$.

3 Merging two triangle-free colours

Notation. In this section $k \geq 3$ and $n \geq 0$ are integers and $\mathbf{t} = (3, 3, k)$, so red and blue are triangle-free. For a $(3, 3, k; n)$ -colouring:

- H is its red-or-blue graph and $G = \overline{H}$ its green graph;
- $h_v = d_1(v) + d_2(v)$ and $z_v = d_3(v) = n - 1 - h_v$ are the degrees of v in H and G ;
- $T(H)$ and $e(H)$ are the numbers of triangles and edges of H ;
- for a vertex set S , $e_H(S) = e_1(S) + e_2(S)$ is the number of edges of H inside S .

By Lemma 2.1, with $D := D_1 = D_2 = R(3, k) - 1$ the common red and blue cap,

$$d_1(v), d_2(v) \leq D, \quad \text{so} \quad h_v \leq 2D, \quad \text{and} \quad z_v \leq D_3 = R(3, 3, k - 1) - 1.$$

Call an integer h *admissible* (as an H -degree) if

$$\max(0, n - 1 - D_3) \leq h \leq \min(n - 1, 2D).$$

Thus every h_v is admissible.

A number $m_k(n)$ is a *valid lower bound* if $e(H) \geq m_k(n)$ in every $(3, 3, k; n)$ -colouring. We use this notion also for $k = 2$, where $m_2(n) = \binom{n}{2}$ is valid, as a $(3, 3, 2; n)$ -colouring has no green edge.

The lemmas below are also applied with $k - 1$ and an order z in place of k and n , for instance to the colouring on a green neighbourhood. Then D , D_3 , admissibility and the functions below change accordingly.

Outline. We prove two inequalities,

$$\sum_v F(h_v) \leq 0 \quad \text{and} \quad \sum_v \Psi(h_v) \leq 0,$$

in which the colouring enters only through the H -degrees (Lemmas 3.2 and 3.3). Both count paths with two edges in H and bound $T(H)$ above by mixed caps (Lemma 3.1).

- The first also uses the cap C_{33} on common green neighbourhoods.
- The second counts the vertex triples spanning one or two edges of H . It needs instead a valid lower bound on the number of H -edges inside each green neighbourhood $N_3(v)$, which carries a $(3, 3, k - 1)$ -colouring.

Lemma 3.4 turns either inequality into a contradiction, excluding the order n , or into a valid lower bound $m_k(n)$, of the kind the second inequality needs at $k + 1$. Lemma 3.5 carries such bounds to larger orders.

The apex bound. In a $(3, 3, k; n)$ -colouring, every triangle of H has two edges of one of red and blue and one of the other, so $T(H) = T_{1,2} + T_{2,1}$. Counting each triangle at its apex by Lemma 2.4, and applying Corollary 2.2 with the mixed caps $C_{12} = C_{21} = R(2, 2, k) - 1 = k - 1$, gives

$$T(H) = \sum_v (e_2(N_1(v)) + e_1(N_2(v))) \leq \sum_v \left(\left\lfloor \frac{(k-1)d_1(v)}{2} \right\rfloor + \left\lfloor \frac{(k-1)d_2(v)}{2} \right\rfloor \right).$$

Lemma 3.1 (Apex bound). *For $0 \leq h \leq 2D$ let $A_k(h) = \lfloor (k-1)h/2 \rfloor$, except that $A_k(2D) = (k-1)D - 1$ when k is even and D is odd. Then every $(3, 3, k; n)$ -colouring satisfies $T(H) \leq \sum_v A_k(h_v) \leq \frac{k-1}{2} \sum_v h_v$.*

Proof. As $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$, the summand at v in the display above is at most $\lfloor (k-1)h_v/2 \rfloor$. If $h_v = 2D$, the caps force $d_1(v) = d_2(v) = D$. If also k is even and D is odd, then $(k-1)D$ is odd, and the summand is $2\lfloor (k-1)D/2 \rfloor = (k-1)D - 1$. So the summand is at most $A_k(h_v)$, and the second inequality holds as $A_k(h) \leq (k-1)h/2$. $\square$

Write

$$p_k(h) = 3(k-1)h - 6A_k(h),$$

six times the gap between the two bounds of Lemma 3.1 at a vertex with $h_v = h$. Thus $p_k = 0$ for odd k . For even k , $p_k(h) = 3$ for odd h , $p_k(2D) = 6$ if D is odd, and $p_k(h) = 0$ otherwise.

The counts below involve $6T(H)$. Six times the first inequality of Lemma 3.1 reads

$$6T(H) \leq \sum_v (3(k-1)h_v - p_k(h_v)). \quad (2)$$

As $p_k \geq 0$, the term $p_k(h_v)$ may be dropped here, and hence the term $p_k(h)$, shown in brackets, from the functions of Lemmas 3.2 and 3.3. The hand proofs of Section 4 drop it, and the procedure of Section 5 (the *ladder*) keeps it.

Lemma 3.2 (Common-neighbour inequality). *Let $U \geq R(3, 3, k - 2)$. Every $(3, 3, k; n)$ -colouring satisfies $\sum_v F(h_v) \leq 0$, where*

$$F(h) = F_{n,k,U}(h) = 3h^2 + (U + 5 - 3n - 3k)h + (n - U - 1)(n - 1) \left[+ p_k(h) \right].$$

With the bracket, F depends also on D , through p_k .

Proof. Step 1: the cherry identity. A vertex v is the middle vertex of $\binom{h_v}{2}$ cherries, paths with two edges in H . For vertices x, y let $c_H(x, y)$ be the number of their common H -neighbours. The endpoints of a cherry are joined either in H , closing a triangle of H , which contains three cherries, or by a green edge xy , and exactly $c_H(x, y)$ cherries have endpoints x and y . So

$$\sum_v \binom{h_v}{2} = 3T(H) + \sum_{xy \in E(G)} c_H(x, y).$$

Step 2: common neighbours across a green edge. For a green edge xy , the H -neighbourhoods of x and y lie among the other $n - 2$ vertices, and those in neither H -neighbourhood form $N_3(x) \cap N_3(y)$. By Lemma 2.1 with $i = j = 3$,

$$|N_3(x) \cap N_3(y)| \leq C_{33} = R(3, 3, k - 2) - 1 \leq U - 1,$$

so inclusion-exclusion gives

$$c_H(x, y) = h_x + h_y - (n - 2) + |N_3(x) \cap N_3(y)| \leq h_x + h_y + U + 1 - n. \quad (3)$$

Step 3: summing over the green edges. Sum (3) over the green edges, using

$$\sum_{xy \in E(G)} (h_x + h_y) = \sum_v h_v z_v \quad \text{and} \quad |E(G)| = \frac{1}{2} \sum_v z_v,$$

double the cherry identity, and bound $6T(H)$ by (2):

$$\sum_v h_v(h_v - 1) \leq \sum_v (3(k - 1)h_v - p_k(h_v) + 2h_v z_v + (U + 1 - n)z_v).$$

Step 4: collecting terms. Moving everything to the left and substituting $z_v = n - 1 - h_v$ turns the summand at v into $F(h_v)$. The coefficients of h_v^2 and h_v are

$$1 + 2 = 3 \quad \text{and} \quad -1 - 3(k - 1) - 2(n - 1) + (U + 1 - n) = U + 5 - 3n - 3k,$$

and the constant term is $-(U + 1 - n)(n - 1)$. □

Lemma 3.3 (Density inequality). *For each admissible h , let $m_{k-1}(n - 1 - h)$ be a valid lower bound for $(3, 3, k - 1; n - 1 - h)$ -colourings. Then every $(3, 3, k; n)$ -colouring satisfies $\sum_v \Psi(h_v) \leq 0$, where*

$$\Psi(h) = \Psi_{n,k}(h) = h(2h - n - 3(k - 1)) + 2m_{k-1}(n - 1 - h) \left[+ p_k(h) \right].$$

Here Ψ depends also on the chosen m_{k-1} and, with the bracket, on D .

Proof. Step 1: two counting identities. Let P_2 and P_1 be the numbers of vertex triples spanning exactly two, respectively exactly one, edges of H . A triple spanning two edges of H contains exactly one cherry, whose endpoints are joined in green. So the cherry identity in the proof of Lemma 3.2 reads

$$\sum_v \binom{h_v}{2} = 3T(H) + P_2.$$

There are $h_v z_v$ corners at v with one edge in H and one in G , and a triple contains two such corners if it spans one or two edges of H , and none otherwise. So

$$\sum_v h_v z_v = 2P_1 + 2P_2.$$

Twice the first identity minus the second, with $h_v - 1 - z_v = 2h_v - n$, gives

$$6T(H) = \sum_v h_v(2h_v - n) + 2P_1.$$

Step 2: the triples with one edge of H . In a triple with exactly one edge of H , the vertex v opposite that edge is joined to both its ends in green, so the edge lies inside $N_3(v)$. Conversely, each edge of H inside $N_3(v)$ forms such a triple with v . So

$$P_1 = \sum_v e_H(N_3(v)).$$

The colouring on $N_3(v)$ is a $(3, 3, k-1; z_v)$ -colouring, as in the proof of Lemma 2.1, and h_v is admissible, so $e_H(N_3(v)) \geq m_{k-1}(z_v)$.

Step 3: comparison with the apex bound. Hence

$$6T(H) \geq \sum_v (h_v(2h_v - n) + 2m_{k-1}(z_v)),$$

and comparing with (2) gives $\sum_v \Psi(h_v) \leq 0$. □

Lemma 3.4. *Let f be a function such that every $(3, 3, k; n)$ -colouring satisfies $\sum_v f(h_v) \leq 0$.*

- (a) *If $\beta > 0$ and $f(h) \geq \alpha - \beta h$ for every admissible h , then every $(3, 3, k; n)$ -colouring has $e(H) \geq n\alpha/(2\beta)$. Hence $\lceil n\alpha/(2\beta) \rceil$ is a valid lower bound $m_k(n)$.*
- (b) *If $n \geq 1$ and $f(h) > 0$ for every admissible h , then no $(3, 3, k; n)$ -colouring exists.*

Proof. Take a $(3, 3, k; n)$ -colouring. Every h_v is admissible, and $\sum_v h_v = 2e(H)$.

(a) $0 \geq \sum_v f(h_v) \geq \sum_v (\alpha - \beta h_v) = n\alpha - 2\beta e(H)$. Divide by $2\beta > 0$, and round up since $e(H)$ is an integer.

(b) $\sum_v f(h_v)$ is a sum of $n \geq 1$ positive terms, so it is positive, a contradiction. □

Part (a) bounds the average H -degree $2e(H)/n$ below by α/β , the zero of $\alpha - \beta h$. So we want a line below f at the admissible integers whose zero lies as far right as possible. Section 4 uses chords (Lemmas 4.2 and 4.4), and Section 5 edges of the lower convex hull of the points $(h, f(h))$. Part (b) is the finite check, step (iii) of the method.

Lemma 3.5 passes valid lower bounds to larger orders,

- by deleting a vertex, whose H -degree is bounded below as its green degree is capped, or
- by averaging over the vertex subsets of a fixed size.

So one application of Lemma 3.4 can serve several orders.

Lemma 3.5. *Let $n \geq 1$, and for each $n' < n$ let $m_k(n')$ be a valid lower bound. Then each of the following is also a valid lower bound for order n :*

- (a) $m_k(n-1) + \max\{0, n - R(3, 3, k-1)\}$;
- (b) $\lceil m_k(s) n(n-1)/(s(s-1)) \rceil$, for any $2 \leq s < n$.

Proof. Take a $(3, 3, k; n)$ -colouring.

(a) By Lemma 2.1, each vertex v has green degree $n-1-h_v \leq R(3, 3, k-1)-1$, so $h_v \geq \max\{0, n - R(3, 3, k-1)\}$. Deleting any vertex v leaves a $(3, 3, k; n-1)$ -colouring with red-or-blue graph $H-v$, so

$$e(H) = e(H-v) + h_v \geq m_k(n-1) + \max\{0, n - R(3, 3, k-1)\}.$$

(b) Each s -subset S of the vertices induces a $(3, 3, k; s)$ -colouring, so $e(H[S]) \geq m_k(s)$. Summing over the $\binom{n}{s}$ subsets counts each edge of H once for each of the $\binom{n-2}{s-2}$ subsets containing both its ends, so

$$e(H) \geq m_k(s) \binom{n}{s} / \binom{n-2}{s-2} = m_k(s) n(n-1)/(s(s-1)).$$

Round up, as $e(H)$ is an integer. □

4 Three upper bounds by hand

Theorem 4.1 uses Lemma 3.2 alone. Applying the same lemma to $(3, 3, 6; z)$ -colourings, with chords in Lemma 3.4(a), gives valid lower bounds $m_6(z)$ for $79 \leq z \leq 87$ (Lemma 4.2). With these, Lemma 3.3 gives Theorem 4.3 and, with Lemma 3.5(a), the bounds $m_7(z)$ for $124 \leq z \leq 128$ of Lemma 4.4, which give Theorem 4.5 in turn.

By computer, Section 8 lowers the three bounds to 86, 125 and 173 (Theorems 8.4 and 8.5). The certificates of Theorem 8.3 use the bounds proved here as caps.

Theorem 4.1. $R(3, 3, 6) \leq 88$.

Proof. Step 1: the degree cap. Suppose a $(3, 3, 6; 88)$ -colouring exists. Here $D = R(3, 6) - 1 = 17$, so $h_v \leq 2D = 34$.

Step 2: the common-neighbour inequality. Lemma 3.2 with $n = 88$, $k = 6$ and $U = R(3, 3, 4) = 30$, without the bracket, gives

$$F(h) = 3h^2 - 247h + 4959.$$

Step 3: positivity. Writing $x = 34 - h$, we get

$$F = 29 + 43x + 3x^2,$$

which is positive for every $x \geq 0$, that is, for every $h \leq 34$, in particular at every admissible h (the green cap is not needed). Lemma 3.4(b) gives a contradiction. $\square$

Why the argument stops here. The argument stops at 88. At $n = 87$,

$$F_{87,6,30}(h) = 3h^2 - 244h + 4816$$

has $F_{87,6,30}(34) = -12$, or -6 with the bracket $p_6(34) = 6$ (as $34 = 2D$ with D odd). And $h = 34$ is still admissible, as the green degree $86 - 34 = 52$ is within the cap $D_3 = 56$.

Nor does the procedure of Section 5, which also applies the density inequality, exclude 87 (Remark 5.1). The certificates of Section 8 do (Theorem 8.4).

Lower bounds $m_6(z)$ from chords. For $(3, 3, 6; z)$ -colourings, Lemma 3.2 with $n = z$, $k = 6$ and $U = R(3, 3, 4) = 30$, without the bracket, gives $\sum_v F_{z,6,30}(h_v) \leq 0$ with

$$F_{z,6,30}(h) = 3h^2 - (3z - 17)h + (z - 31)(z - 1).$$

As Lemma 3.4(a) needs a line below $F_{z,6,30}$ only at integers, we use a chord rather than a tangent. For an integer q ,

$$\begin{aligned} F_{z,6,30}(h) &= \alpha - \beta h + 3(h - q)(h - q - 1) \geq \alpha - \beta h, \\ \beta &= 3z - 20 - 6q, \quad \alpha = (z - 31)(z - 1) - 3q(q + 1), \end{aligned} \tag{4}$$

where the inequality holds at every integer h , as $(h - q)(h - q - 1)$ is then a product of two consecutive integers and so nonnegative. So if $\beta > 0$, Lemma 3.4(a) shows that $\lceil z\alpha/(2\beta) \rceil$ is a valid lower bound $m_6(z)$.

The chord $\alpha - \beta h$ meets $F_{z,6,30}$ at $h = q$ and $h = q + 1$. At the smaller root r of $F_{z,6,30}$ it takes the value $-3(r - q)(r - q - 1)$, which is positive exactly when $q < r < q + 1$. So for $q = \lfloor r \rfloor$ the chord, if decreasing ($\beta > 0$), crosses zero to the right of r (r is not an integer for $79 \leq z \leq 87$, as the discriminant $(3z - 17)^2 - 12(z - 31)(z - 1)$, which decreases from 3472 to 1744 on this range, is never a perfect square). By contrast, for every other q with $\beta > 0$, and for every decreasing tangent, which is at most $F_{z,6,30}(r) = 0$ at r , the zero lies at or left of r . Thus $q = \lfloor r \rfloor$ gives the strongest of these bounds. The procedure of Section 5 raises some of them (Remark 5.1).

Lemma 4.2. For $79 \leq z \leq 87$,

$$m_6(z) = \lceil z\alpha/(2\beta) \rceil$$

is a valid lower bound for $e(H)$ in $(3, 3, 6; z)$ -colourings, with α and β from (4) for $q = \lfloor r \rfloor$, where r is the smaller root of $F_{z,6,30}$, shown rounded to two decimals:

z	79	80	81	82	83	84	85	86	87
r	26.85	27.63	28.42	29.24	30.08	30.94	31.83	32.75	33.71
q	26	27	28	29	30	30	31	32	33
β	61	58	55	52	49	52	49	46	43
α	1638	1603	1564	1521	1474	1609	1560	1507	1450
$m_6(z)$	1061	1106	1152	1200	1249	1300	1354	1409	1467

Proof. Every β in the table is positive, so (4) and Lemma 3.4(a) apply. $\square$

Theorem 4.3. $R(3, 3, 7) \leq 129$.

Proof. Step 1: the admissible degrees. Suppose a $(3, 3, 7; 129)$ -colouring exists. Here $h_v \leq 2D = 2(R(3, 7) - 1) = 44$, and $z_v = 128 - h_v \leq 87$ by Theorem 4.1. So the admissible H -degrees are $41, \dots, 44$, with green degrees $84, \dots, 87$.

Step 2: the density inequality. Lemma 3.3 with $n = 129$, $k = 7$ and the values $m_6(z)$ of Lemma 4.2 gives $\sum_v \Psi(h_v) \leq 0$ for

$$\Psi(h) = h(2h - 147) + 2m_6(128 - h).$$

The bracket p_7 vanishes, as k is odd.

Step 3: positivity. But Ψ is positive at every admissible h :

h	$z = 128 - h$	$m_6(z)$	$\Psi(h)$
41	87	1467	269
42	86	1409	172
43	85	1354	85
44	84	1300	4

so Lemma 3.4(b) gives a contradiction. $\square$

Comparison. With Theorem 4.1, (1) gives only $23 + 23 + 88 - 1 = 133$ for $R(3, 3, 7)$.

The common-neighbour inequality does not exclude $n = 129$, and neither does the density inequality without a nontrivial lower bound m_6 :

- Lemma 3.2, with $U = 57$ from $R(3, 3, 5) \leq 57$, gives $F(h) = 3h^2 - 346h + 9088$ ($p_7 = 0$), which is decreasing for $h \leq 57$, so its largest value at an admissible h is $F(41) = -55$;
- and Lemma 3.3 with the trivial $m_6 = 0$ gives $\Psi(h) = h(2h - 147) < 0$ at every admissible h .

By Lemma 4.2, in contrast, each green neighbourhood, which carries a $(3, 3, 6; z)$ -colouring with $84 \leq z \leq 87$, has average red-or-blue degree at least $2m_6(z)/z > 30$, close to the cap $2(R(3, 6) - 1) = 34$.

Lemma 4.4. *Valid lower bounds for $e(H)$ in $(3, 3, 7; z)$ -colourings are*

$$\begin{aligned} m_7(124) &= 2495, & m_7(125) &= 2532, & m_7(126) &= 2570, \\ m_7(127) &= 2609 & \text{and} & & m_7(128) &= 2649. \end{aligned}$$

Proof. Step 1: the admissible degrees. Take a $(3, 3, 7; 124)$ -colouring. Here $h_v \leq 2(R(3, 7) - 1) = 44$, and $z_v = 123 - h_v \leq 87$ by Theorem 4.1. So the admissible H -degrees are $36, \dots, 44$, with green degrees $79, \dots, 87$, those covered by Lemma 4.2.

Step 2: the density inequality. Lemma 3.3 with $n = 124$, $k = 7$ and these values $m_6(z)$ gives $\sum_v \Psi(h_v) \leq 0$ for

$$\Psi(h) = h(2h - 142) + 2m_6(123 - h).$$

The bracket p_7 vanishes as k is odd.

Step 3: a chord below Ψ . As Ψ takes both signs, we apply Lemma 3.4(a) with the line $3138 - 78h$, the chord of Ψ through $h = 40$ and $h = 41$, where Ψ changes sign:

h	36	37	38	39	40	41	42	43	44
$\Psi(h)$	414	302	200	104	18	-60	-132	-196	-254
$\Psi(h) - (3138 - 78h)$	84	50	26	8	0	0	6	20	40

The last row is nonnegative, so

$$\Psi(h) \geq 3138 - 78h$$

at every admissible h . Lemma 3.4(a) with $\alpha = 3138$ and $\beta = 78$ then gives

$$e(H) \geq 124 \cdot 3138 / 156 > 2494.$$

So $m_7(124) = 2495$ is valid.

Step 4: the larger orders. For $125 \leq z \leq 128$, Lemma 3.5(a) with $k = 7$ and $R(3, 3, 6) \leq 88$ gives

$$m_7(z) = m_7(z - 1) + (z - 88),$$

adding 37, 38, 39 and 40 in turn. □

Theorem 4.5. $R(3, 3, 8) \leq 179$.

Proof. Step 1: the admissible degrees. Suppose a $(3, 3, 8; 179)$ -colouring exists. Here $h_v \leq 2D = 2(R(3, 8) - 1) = 54$, and $z_v = 178 - h_v \leq 128$ by Theorem 4.3. So the admissible H -degrees are $50, \dots, 54$, with green degrees $124, \dots, 128$, exactly the orders of Lemma 4.4.

Step 2: the density inequality. Lemma 3.3 with $n = 179$, $k = 8$ and these values $m_7(z)$, without the bracket, gives $\sum_v \Psi(h_v) \leq 0$ for

$$\Psi(h) = h(2h - 200) + 2m_7(178 - h).$$

Step 3: positivity. But Ψ is positive at all five admissible h :

h	$z = 178 - h$	$m_7(z)$	$\Psi(h)$
50	128	2649	298
51	127	2609	220
52	126	2570	148
53	125	2532	82
54	124	2495	22

So Lemma 3.4(b) gives a contradiction. $\square$

5 The ladder: $R(3, 3, k)$ for $9 \leq k \leq 15$

The procedure of this section, the *ladder*, automates Section 4, in levels $k = 3, 4, \dots, 15$, each level using the bounds and tables of the levels below it. Write B_k for the upper bound on $R(3, 3, k)$ that it uses. Level k records valid lower bounds $m_k(n)$ for $2 \leq n \leq B_k - 1$. For $k \leq 5$, B_k is an input: $B_1 = R(3, 3, 1) = 1$ and $B_2 = R(3, 3, 2) = 6$ are degenerate values (Section 2), and $B_3 = 17$, $B_4 = 30$ and $B_5 = 57$, from Table 3, are the only three-colour inputs. For $k \geq 6$, B_k is the first order excluded at level k , where the level stops; so $R(3, 3, k) \leq B_k$.

Each level. The procedure starts from the trivial table $m_2(s) = \binom{s}{2}$ for $s \leq 5$ (Section 3), and puts $m_k(0) = m_k(1) = 0$. Level k uses:

- the caps $D = r_k - 1$, where r_k is the value or bound for $R(3, k)$ in Table 3 (so $D = 40, 49, 58, 67, 76, 86$ for $k = 10, \dots, 15$), and $D_3 = B_{k-1} - 1$;
- the value $U = B_{k-2}$ in F (Lemma 3.2);
- and in Ψ (Lemma 3.3) the table m_{k-1} , which covers every green degree $z \leq D_3$.

Both F and Ψ include the bracket p_k .

The level treats the orders $n = 2, 3, \dots$ from the smallest upwards, so that Lemma 3.5 can use the values of m_k at smaller orders. Levels 3, 4 and 5 run through all $n < B_k$ and exclude none (Remark 5.1). They serve to build m_3 , m_4 and m_5 , the last of which enters Ψ at level 6.

Treatment of one order. At order n , the admissible H -degrees are the integers h with

$$h_{\min} \leq h \leq \min(2D, n - 1),$$

where $h_{\min} = \max(0, n - B_{k-1})$.

- If F or Ψ is positive at every admissible h , then n is excluded by Lemma 3.4(b).
- Otherwise, for each $f \in \{F, \Psi\}$ let h_f^* be the least real $h \geq h_{\min}$ at which the lower convex hull of the points $(h, f(h))$, h admissible, viewed as a piecewise-linear function on $[h_{\min}, \min(2D, n - 1)]$, is at most 0. Then $\lceil nh_f^*/2 \rceil$ is a valid lower bound.
 - If $h_f^* = h_{\min}$, that is, $f(h_{\min}) \leq 0$, this holds because every H -degree is at least $h_{\min}$.
 - Otherwise the hull edge reaching $(h_f^*, 0)$ from the left lies on a line $\alpha - \beta h$ with $\beta > 0$ and zero at h_f^* . This line lies below the hull, so it is a *supporting line*:

$$f(h) \geq \alpha - \beta h$$

at every admissible h . Lemma 3.4(a) then gives $\lceil n\alpha/(2\beta) \rceil = \lceil nh_f^*/2 \rceil$, as $\alpha/\beta = h_f^*$ (no supporting line with $\beta > 0$ does better, as each lies below the hull and so is at most 0 at h_f^*).

- (iii) The recorded $m_k(n)$ is the largest of the bounds from F and Ψ and those that Lemma 3.5(a) and (b) give from the values at smaller orders, with B_{k-1} in place of $R(3, 3, k-1)$ in (a).

Remark 5.1 (Sanity checks). The levels $k \leq 8$ of the procedure behave as follows.

For $k = 3$ and $k = 4$ it excludes no order below the exact values $R(3, 3, 3) = 17$ and $R(3, 3, 4) = 30$. As colourings exist at all these orders, this tests soundness.

At $n = 16$ the caps $D = R(3, 3) - 1 = 5$ and $D_3 = R(3, 3, 2) - 1 = 5$ give $h_v \leq 10$ and $h_v = 15 - z_v \geq 10$. So $h_v = 10$ at every vertex, and $e(H) = 80$, in every $(3, 3, 3; 16)$ -colouring, such as that of Greenwood and Gleason [13], in which each colour class is the 5-regular Clebsch graph. Lemmas 3.2 and 3.3, with $U = R(3, 3, 1) = 1$ and the valid $m_2(s) = \binom{s}{2}$ (a $(3, 3, 2)$ -colouring has no green edge), give

$$F_{16,3,1}(h) = 3h^2 - 51h + 210 \quad \text{and} \quad \Psi_{16,3}(h) = h(2h - 22) + 2 \binom{15 - h}{2},$$

with $p_3 = 0$, and both vanish at $h = 10$. So neither inequality excludes $n = 16$, and the procedure records the exact value $m_3(16) = 80$. This tests the correctness of the inequalities rather than their strength.

For $k = 5$ it excludes no order below 57, the least order at which the caps alone leave no admissible h , so it does not improve $R(3, 3, 5) \leq 57$.

For $k = 6, 7, 8$ it reaches exactly the hand-proved bounds 88, 129 and 179, excluding no smaller order, although it uses more than those proofs do:

- the bracket p_k ;
- both inequalities at every order;
- supporting lines from lower convex hulls;
- and Lemma 3.5(b).

These raise some values $m_k(z)$ above those of the hand proofs, but not the bounds.

Theorem 5.2 (computer-assisted). *The procedure gives*

$$(B_9, \dots, B_{15}) = (247, 315, 407, 512, 628, 756, 901).$$

Hence $R(3, 3, k) \leq B_k$ for $9 \leq k \leq 15$. These bounds form the first row of Table 4.

The run recomputes $B_6 = 88$, $B_7 = 129$ and $B_8 = 179$, agreeing with Section 4, so the theorem does not rest on the hand proofs there. Its external inputs, the values and bounds for $R(3, k)$ and B_3, B_4, B_5 from Table 3, are all unstarred, so it uses no unpublished bound.

For $9 \leq k \leq 15$, it is Ψ that excludes B_k by Lemma 3.4(b): its minimum over the admissible h at $n = B_k$, in the last row of Table 4, is positive, whereas F is negative at $h = 2D$, the largest admissible h . In every column this minimum is attained at $h = 2D$, where $p_k(2D) = 0$ (for odd k as $p_k = 0$, and for $k = 10, 12, 14$ as $D = 40, 58$ and 76 are even), so it equals

$$2D(4D - B_k - 3(k - 1)) + 2m_{k-1}(z), \quad z = B_k - 1 - 2D.$$

For $k = 9, \dots, 15$ the green degree is $z = 176, 234, 308, 395, 493, 603, 728$, and $m_{k-1}(z) = 4599, 7317, 11810, 18216, 26598, 37421, 51609$.

The run writes a certificate, which a checker that shares no code with the generating program verifies in exact rational arithmetic. With valid lower bounds and exclusions from the certificates of Section 8, the same procedure gives the smaller bounds of Table 5 (Theorem 8.5).

k	9	10	11	12	13	14	15
Theorem 5.2	247	315	407	512	628	756	901
recurrence from 91 [†]	261	342	441	558	693	846	1019
recurrence from B_{k-1}	249	328	413	524	647	781	929
admissible h	68–70	68–80	92–98	105–116	116–134	128–152	145–172
minimum of Ψ	28	74	2	124	132	210	190

Table 4: The ladder. The row “recurrence from 91” gives the daggered benchmarks of Table 1: (1) with parity, iterated from $R(3, 3, 6) \leq 91$ [20] through 136 and 191 for $k = 7, 8$. The row “recurrence from B_{k-1} ” applies (1) with parity once, to B_{k-1} . The last two rows concern the excluded order $n = B_k$: its admissible H -degrees, and the minimum over them of Ψ , with the bracket p_k and the table m_{k-1} recorded at level $k - 1$. The tables m_{k-1} are not reproduced. The programs for Theorem 5.2 (Section 10) regenerate them.

Verification of Theorem 5.2. The certificate. The certificate has one row for each level k with $3 \leq k \leq 15$ and each order $2 \leq n \leq B_k - 1$, so $\sum_{k=3}^{15} (B_k - 2) = 4240$ rows. Each row states a claimed value of $m_k(n)$ and, for each $f \in \{F, \Psi\}$, certifies the bound from f :

- either it records the admissible ends $h' < h''$ of the hull edge used, so that the supporting line $\alpha - \beta h$ is the line through $(h', f(h'))$ and $(h'', f(h''))$;
- or it records that $f(h_{\min}) \leq 0$, and the bound is then $\lceil nh_{\min}/2 \rceil$, which needs no line, as every H -degree is at least $h_{\min}$.

The checker. The checker treats the rows in the order of the procedure, in exact rational arithmetic. For each recorded line it evaluates f from its definition, with the bracket p_k and, in Ψ , the values m_{k-1} it has already verified. It then checks the hypotheses of Lemma 3.4(a): $\beta > 0$, and $f(h) \geq \alpha - \beta h$ at every admissible h . Summed over the rows, there are 420,545 admissible degrees. It accepts the claimed $m_k(n)$ only if it is at most the largest of the two certified bounds and those that Lemma 3.5(a) and (b) give from the values it has verified at smaller orders.

The bounds B_k . Finally, for $k \leq 5$ it checks that B_k is the input, and for $6 \leq k \leq 15$ that at $n = B_k$, for which the certificate states no $m_k(n)$ but names the excluding inequality, that inequality is positive at every admissible h , so that Lemma 3.4(b) excludes B_k . $\square$

6 Weighted triangle identities

In this section $t_1 = 3$ and $t_2, t_3 \geq 4$, so red is the only colour required to be triangle-free, and $M_1 = 0$. This covers $R(3, 4, 5)$, $R(3, 4, 6)$ and $R(3, 5, 5)$, and the three numbers of Proposition 6.5. Theorem 8.3 improves all six bounds of this section by computer. The arguments here can be checked by hand.

The merging of Section 3 needs two triangle-free colours, so here we keep the three colours apart and take a weighted sum of the corner identities of Lemma 2.4.

Lemma 6.1. *For real weights $w_1, w_2, w_3, w_{12}, w_{13}, w_{23}$, not necessarily positive, put $w_{ji} := w_{ij}$ and*

$$W(\mathbf{d}) = \sum_i w_i \binom{d_i}{2} + \sum_{i < j} w_{ij} d_i d_j.$$

Then for every 3-colouring of K_n ,

$$\sum_v W(\mathbf{d}(v)) = \sum_i 3w_i M_i + \sum_{i \neq j} (w_i + 2w_{ij}) T_{i,j} + (w_{12} + w_{13} + w_{23}) X.$$

Proof. In Lemma 2.4, multiply the identity for $\sum_v \binom{d_i(v)}{2}$ by w_i and that for $\sum_v d_i(v) d_j(v)$ by w_{ij} , and sum over all i and all pairs $i < j$. The first gives $3w_i$ to M_i and w_i to each $T_{i,l}$ with $l \neq i$. The second, for the pair $i < j$, gives $2w_{ij}$ to both $T_{i,j}$ and $T_{j,i}$, and w_{ij} to X . Collect terms, using $w_{ji} = w_{ij}$. $\square$

The types and the sign conditions. Since $M_1 = 0$, nine types remain: M_2, M_3 , the six $T_{i,j}$ and X . All but X are bounded above by Lemma 2.4 and Corollary 2.2:

$$3M_i \leq \sum_v \lfloor C_{ii} d_i(v)/2 \rfloor \quad \text{for } i = 2, 3 \quad \text{and} \quad T_{i,j} \leq \sum_v \lfloor C_{ij} d_i(v)/2 \rfloor \quad \text{for } i \neq j.$$

The type X is bounded only below, by 0.

We impose the *sign conditions*:

- $w_{12} + w_{13} + w_{23} = 0$;
- $w_2, w_3 \geq 0$; and
- $w_i + 2w_{ij} \geq 0$ for all $i \neq j$.

A negative coefficient of X would also do, but we do not need one. In Lemma 6.1 the sign conditions make the coefficient of X vanish and those of M_2, M_3 and the six $T_{i,j}$ nonnegative. The coefficient $3w_1$ of $M_1 = 0$ may have either sign.

The function Φ . Multiplying the bounds on $3M_2, 3M_3$ and $T_{i,j}$ by w_2, w_3 and $w_i + 2w_{ij}$ then gives $\sum_v \Phi(\mathbf{d}(v)) \leq 0$, step (ii) of the method, for

$$\Phi(\mathbf{d}) = W(\mathbf{d}) - \sum_{i=2,3} w_i \left\lfloor \frac{C_{ii} d_i}{2} \right\rfloor - \sum_{i \neq j} (w_i + 2w_{ij}) \left\lfloor \frac{C_{ij} d_i}{2} \right\rfloor.$$

Dropping the floors only lowers Φ , so the inequality also holds for the floor-free form, which the hand proofs use. Types with coefficient zero drop out of Φ and need no cap.

Step (iii) needs weights for which Φ is positive at every admissible profile. At each profile Φ is linear in the weights, so these requirements and the sign conditions form a finite system of linear inequalities in the six weights. We give explicit solutions, and the proofs check them directly.

Theorem 6.2. $R(3, 4, 5) \leq 155$.

Proof. The colour-degree caps. Suppose a $(3, 4, 5; 155)$ -colouring exists, and write $(r, b, g) = \mathbf{d}(v)$ for a vertex v , so that $r + b + g = 154$. By Lemma 2.1,

$$r \leq D_1 = R(2, 4, 5) - 1 = R(4, 5) - 1 = 24 \quad \text{and} \quad b \leq D_2 = R(3, 3, 5) - 1 \leq 56.$$

Choosing which types vanish. We choose the weights so that the coefficients of X , $T_{1,3}$, $T_{2,1}$ and $T_{3,1}$ in Lemma 6.1 vanish. Writing $w_{12} = -\alpha$ and $w_{13} = -\gamma$, this forces $w_1 = w_3 = 2\gamma$, $w_2 = 2\alpha$ and $w_{23} = \alpha + \gamma$, so that

$$W = 2\gamma \binom{r}{2} + 2\alpha \binom{b}{2} + 2\gamma \binom{g}{2} - \alpha r b - \gamma r g + (\alpha + \gamma) b g.$$

As $M_1 = 0$, Lemma 6.1 then gives

$$\sum_v W = 2(\gamma - \alpha)T_{1,2} + (4\alpha + 2\gamma)T_{2,3} + (2\alpha + 4\gamma)T_{3,2} + 2\alpha \cdot 3M_2 + 2\gamma \cdot 3M_3.$$

The sign conditions hold exactly when $\gamma \geq \alpha \geq 0$.

Capping the surviving types. By Lemma 2.1, the surviving types have caps

$$\begin{aligned} C_{12} &= R(2, 3, 5) - 1 = 13 & C_{22} &= R(3, 2, 5) - 1 = 13 \\ \text{and} \quad C_{23} &= C_{32} = C_{33} = R(3, 3, 4) - 1 = 29. \end{aligned}$$

So Corollary 2.2 and Lemma 2.4 give, without floors,

$$T_{1,2} \leq \frac{13}{2} \sum_v r, \quad T_{2,3} \leq \frac{29}{2} \sum_v b, \quad T_{3,2} \leq \frac{29}{2} \sum_v g, \quad 3M_2 \leq \frac{13}{2} \sum_v b, \quad 3M_3 \leq \frac{29}{2} \sum_v g.$$

Multiplying these by the coefficients above gives $\sum_v \Phi(\mathbf{d}(v)) \leq 0$ for the floor-free form of Φ , here

$$\Phi = W - 13(\gamma - \alpha)r - (71\alpha + 29\gamma)b - (29\alpha + 87\gamma)g.$$

Reduction to two variables. Put $x = 24 - r \geq 0$ and $y = 56 - b \geq 0$, so that $(r, b, g) = (24 - x, 56 - y, 74 + x + y)$. Then

$$\Phi = (70\alpha - 52\gamma) + (70\alpha + 132\gamma)x + 3\gamma x^2 + (47\gamma - 63\alpha)y + 2(\gamma - \alpha)xy.$$

Indeed, its three parts are found as follows.

- *Constant term.* At $(24, 56, 74)$ we have $W = 5880\alpha + 8322\gamma$, and the subtracted terms total $5810\alpha + 8374\gamma$.
- *Linear part.* The partial derivatives of Φ there in r , b , g are $-43\alpha - 40\gamma$, $90\alpha + 45\gamma$ and $27\alpha + 92\gamma$, as $\binom{d}{2}$ has derivative $d - \frac{1}{2}$. So the linear part is $(70\alpha + 132\gamma)x + (47\gamma - 63\alpha)y$.

- *Quadratic part.* The quadratic part $\gamma r^2 + \alpha b^2 + \gamma g^2 - \alpha r b - \gamma r g + (\alpha + \gamma) b g$ of W , at the displacement $(-x, -y, x + y)$, is $3\gamma x^2 + 2(\gamma - \alpha)xy$.

Fixing the weights. We take $\alpha = 29$ and $\gamma = 39$, that is,

$$(w_1, w_2, w_3; w_{12}, w_{13}, w_{23}) = (78, 58, 78; -29, -39, 68).$$

The choice is explained after the proof. Then

$$\sum_v W = 20T_{1,2} + 194T_{2,3} + 214T_{3,2} + 58 \cdot 3M_2 + 78 \cdot 3M_3,$$

so the sign conditions hold. Here $\Phi = W - 130r - 3190b - 4234g$, and

$$\Phi = 2 + 7178x + 117x^2 + 6y + 20xy \geq 2.$$

So $\sum_v \Phi(\mathbf{d}(v)) \geq 2 \cdot 155 > 0$, a contradiction. $\square$

The choice of α and γ . Given $\gamma \geq \alpha \geq 0$, the expression for Φ in the proof is positive on the whole quadrant $x, y \geq 0$ exactly when $70\alpha > 52\gamma$ and $47\gamma \geq 63\alpha$, that is,

$$63/47 \leq \gamma/\alpha < 35/26.$$

This interval has width $7/1222$, which explains the thin margin in the proof. The fraction in it with the least denominator is $39/29$. With the green cap $g \leq R(3, 4, 4) - 1 \leq 76$, that is $x + y \leq 2$, the lower end would come instead from $\Phi(0, 2) = 14(3\gamma - 4\alpha) > 0$. This would widen the interval only to $4/3 < \gamma/\alpha < 35/26$, of width $1/78$.

The endpoint $\gamma/\alpha = 4/3$. The proof of Theorem 6.2 needs neither the green cap nor a divisibility argument like parity. The weights at the endpoint $\gamma/\alpha = 4/3$, which the interval above leaves out, also prove it, but only with both. For $\alpha = 3$ and $\gamma = 4$,

$$\Phi = 2 + 738x + 12x^2 - y + 2xy,$$

which is -2 at $(x, y) = (0, 4)$, with or without the floors, as r, b and g are then even. Under the green cap $x + y \leq 2$, however:

- if $x = 0$, then $\Phi = 2 - y \geq 0$;
- if $x \geq 1$, then $\Phi \geq 752$, as $y(2x - 1) \geq 0$.

So $\Phi \geq 0$, with equality only at $(x, y) = (0, 2)$, and $\sum_v \Phi(\mathbf{d}(v)) \leq 0$ forces every vertex to have $(r, b, g) = (24, 54, 76)$.

A divisibility argument modulo 3, an analogue of the parity argument, excludes this. For a vertex v and $u \in N_3(v)$, Lemma 2.1 caps the numbers of red, blue and green neighbours of u in $N_3(v)$ by

$$\begin{aligned} C_{31} &= R(2, 4, 4) - 1 = R(4, 4) - 1 = 17 \\ C_{32} &= R(3, 3, 4) - 1 = 29 \quad \text{and} \quad C_{33} = R(3, 4, 3) - 1 = 29. \end{aligned}$$

These sum to $75 = d_3(v) - 1$, so all are attained and $e_3(N_3(v)) = 76 \cdot 29/2 = 1102$. So $3M_3 = 155 \cdot 1102 = 170810$ by Lemma 2.4, which is not divisible by 3, a contradiction.

The weights of L . For Theorems 6.3 and 6.4, and for $R(3, 5, 6)$ in Proposition 6.5, we require the coefficients of X , $T_{1,2}$, $T_{1,3}$, $T_{2,3}$ and $T_{3,2}$ to vanish. This forces

$$w_{12} = w_{13} = -w_1/2 \quad w_{23} = w_1 \quad \text{and} \quad w_2 = w_3 = -2w_1,$$

which fixes the weights up to a common factor. We take

$$(w_1, w_2, w_3; w_{12}, w_{13}, w_{23}) = (-2, 4, 4; 1, 1, -2).$$

With $(r, b, g) = \mathbf{d}(v)$, write L for the resulting function W . Then Lemma 6.1 and $M_1 = 0$ give

$$\begin{aligned} L &= -2\binom{r}{2} + 4\binom{b}{2} + 4\binom{g}{2} + rb + rg - 2bg, \\ \sum_v L(\mathbf{d}(v)) &= 6T_{2,1} + 6T_{3,1} + 12M_2 + 12M_3. \end{aligned} \tag{5}$$

The remaining coefficients $6 = w_2 + 2w_{12} = w_3 + 2w_{13}$ and $12 = 3w_2 = 3w_3$ are positive, so the sign conditions hold. The four surviving types have caps C_{21} , C_{22} , C_{31} and C_{33} . So the terms subtracted from L involve only b and g , and the blue-green cap C_{23} used in Theorem 6.2 is not needed.

Each floor-free Φ below is L minus a linear function. So its quadratic part at a displacement $\delta = (\delta_r, \delta_b, \delta_g)$ is that of L , namely

$$-\delta_r^2 + 2\delta_b^2 + 2\delta_g^2 + \delta_r\delta_b + \delta_r\delta_g - 2\delta_b\delta_g.$$

Between profiles with the same sum, $\delta_r = -\delta_b - \delta_g$, and this equals $-6\delta_b\delta_g$. Hence

$$\Phi(\mathbf{d} + \delta) = \Phi(\mathbf{d}) + \nabla\Phi(\mathbf{d}) \cdot \delta - 6\delta_b\delta_g \quad \text{if } \delta_r + \delta_b + \delta_g = 0,$$

where $\nabla\Phi$ is computed with $d - \frac{1}{2}$ as the derivative of $\binom{d}{2}$. So each proof below needs only a value and a gradient of Φ .

Theorem 6.3. $R(3, 4, 6) \leq 276$.

Proof. The colour-degree caps. Suppose a $(3, 4, 6; 276)$ -colouring exists, and write $(r, b, g) = \mathbf{d}(v)$, so that $r + b + g = 275$. By Lemma 2.1:

- $r \leq D_1 = R(2, 4, 6) - 1 = R(4, 6) - 1 \leq 40$, by the published bound $R(4, 6) \leq 41$ rather than the starred $R(4, 6) \leq 40$; and
- $b \leq D_2 = R(3, 3, 6) - 1 \leq 87$ by Theorem 4.1.

No cap on g is needed, so Theorem 6.2 is not used.

Capping the surviving types. The four types in (5) have caps

$$\begin{aligned} C_{21} &= R(2, 3, 6) - 1 = R(3, 6) - 1 = 17 & C_{22} &= R(3, 2, 6) - 1 = 17 \\ C_{31} &= R(2, 4, 5) - 1 = R(4, 5) - 1 = 24 & \text{and} & & C_{33} &= R(3, 4, 4) - 1 \leq 76. \end{aligned}$$

So Corollary 2.2 and Lemma 2.4, with the floors dropped, give

$$\begin{aligned} T_{2,1} &\leq \frac{17}{2} \sum_v b & 3M_2 &\leq \frac{17}{2} \sum_v b \\ T_{3,1} &\leq 12 \sum_v g & \text{and} & \quad 3M_3 \leq 38 \sum_v g. \end{aligned}$$

Multiplying these by their coefficients 6, 4, 6 and 4 in (5), where $12M_i = 4 \cdot 3M_i$, gives $\sum_v \Phi(\mathbf{d}(v)) \leq 0$ for the floor-free form of Φ , here

$$\Phi = L - 85b - 224g,$$

with $85 = 6 \cdot \frac{17}{2} + 4 \cdot \frac{17}{2}$ and $224 = 6 \cdot 12 + 4 \cdot 38$.

Reduction to two variables. Put $x = 40 - r \geq 0$ and $y = 87 - b \geq 0$, so that $g = 148 + x + y$. At $(r, b, g) = (40, 87, 148)$, $L = 40564$ and $85b + 224g = 40547$, so $\Phi = 17$, and $\nabla \Phi = (156, 5, 232)$. The expansion after (5) with $\delta = (-x, -y, x + y)$ gives

$$\Phi = 17 + 76x + 227y + 6xy + 6y^2 \geq 17.$$

So $\sum_v \Phi(\mathbf{d}(v)) \geq 17 \cdot 276 > 0$, a contradiction. $\square$

Why it stops here. With the starred $R(4, 6) \leq 40$, the cap $r \leq 39$ gives $x \geq 1$ in the proof, and the minimum of Φ rises to $17 + 76 = 93$. Even so, these weights do not exclude $n = 275$. The admissible profile $(39, 87, 148)$ is $(40, 87, 148)$ with r lowered by 1, so $\Phi = 17 - 156 - 1 = -140$ (-1 being the coefficient of r^2 in L). With the floors it is -135 , as $17b$ is odd.

Theorem 6.4. $R(3, 5, 5) \leq 346$.

Proof. The caps. Suppose a $(3, 5, 5; 346)$ -colouring exists, and write $(r, b, g) = \mathbf{d}(v)$, so that $r + b + g = 345$. By Lemma 2.1, $r \leq D_1 = R(2, 5, 5) - 1 = R(5, 5) - 1 \leq 45$ by $R(5, 5) \leq 46$. As $t_2 = t_3$, blue and green play symmetric roles:

- $b, g \leq D_2 = D_3 = R(3, 4, 5) - 1 \leq 154$ by Theorem 6.2; and
- the four types in (5) have caps $C_{21} = C_{31} = R(2, 4, 5) - 1 = R(4, 5) - 1 = 24$ and $C_{22} = C_{33} = R(3, 3, 5) - 1 \leq 56$.

Capping the surviving types. As in Theorem 6.3, without floors,

$$T_{i,1} \leq 12 \sum_v d_i(v) \quad \text{and} \quad 3M_i \leq 28 \sum_v d_i(v) \quad \text{for } i = 2, 3.$$

Multiplying these by 6 and 4, respectively, gives $\sum_v \Phi(\mathbf{d}(v)) \leq 0$ for the floor-free form of Φ , here

$$\Phi = L - 184(b + g),$$

with $184 = 6 \cdot 12 + 4 \cdot 28$.

Reduction to two variables. Put $x = 154 - b \geq 0$ and $y = 154 - g \geq 0$, so that $r = 37 + x + y$, and $r \leq 45$ gives $x + y \leq 8$. At $(r, b, g) = (37, 154, 154)$, $L = 56880 = 184(b + g) + 208$, so $\Phi = 208$, and $\nabla\Phi = (235, 159, 159)$. The expansion after (5) with $\delta = (x + y, -x, -y)$ gives

$$\Phi = 208 + 76x + 76y - 6xy.$$

For fixed $y \leq 8$, Φ is increasing in x , as $76 - 6y \geq 28$, so $\Phi \geq 208 + 76y \geq 208$. Hence $\sum_v \Phi(\mathbf{d}(v)) \geq 208 \cdot 346 > 0$, a contradiction. $\square$

Why it stops here. These weights do not exclude $n = 345$: the admissible profile $(36, 154, 154)$ gives $\Phi = -28$, with or without the floors, as the caps 24 and 56 are even.

The gain over the recurrence. For comparison, (1) with our hand bounds 88 and 155 in place of 91 and 157 gives

$$41 + 88 + 155 - 1 = 283 \quad \text{for } R(3, 4, 6)$$

$$46 + 155 + 155 - 1 = 355 \quad \text{for } R(3, 5, 5).$$

Both are odd, so parity does not apply. So from the same colour-degree caps counting gains 7 and 9, and Theorem 6.3 needs only the red and blue ones. With the starred $R(4, 6) \leq 40$, the recurrence gives 282, lowered by parity to 281 as 282 and 88 are even, and the gain is 5.

Proposition 6.5. $R(3, 4, 7) \leq 456$, $R(3, 4, 8) \leq 700$ and $R(3, 5, 6) \leq 683$. Without the starred inputs of Table 3, $R(3, 4, 7) \leq 458$, $R(3, 4, 8) \leq 705$ and $R(3, 5, 6) \leq 685$.

Proof. The weights. For $R(3, 5, 6)$ we use the weights of L . For $\mathbf{t} = (3, 4, k)$ with $k = 7, 8$ we use

$$(w_1, w_2, w_3; w_{12}, w_{13}, w_{23}) = (2, 2, 2; 2, -1, -1).$$

Here $w_{12} + w_{13} + w_{23} = 0$ and $w_i + 2w_{i3} = w_3 + 2w_{3i} = 0$ for $i = 1, 2$, while $w_1 + 2w_{12} = w_2 + 2w_{12} = 3w_2 = 3w_3 = 6$. So Lemma 6.1 and $M_1 = 0$ give

$$W = 2\binom{r}{2} + 2\binom{b}{2} + 2\binom{g}{2} + 2rb - rg - bg,$$

$$\sum_v W(\mathbf{d}(v)) = 6T_{1,2} + 6T_{2,1} + 6M_2 + 6M_3.$$

The caps and the function Φ . Of these four types only M_3 is counted in green neighbourhoods, with cap $C_{33} = R(3, 4, k - 2) - 1$. The other three have the common cap

$$C := C_{12} = C_{21} = C_{22} = R(3, k) - 1.$$

With the weights of L , the type $T_{3,1}$ would take the place of $T_{1,2}$, bringing in the larger cap $C_{31} = R(4, k - 1) - 1$ multiplied by the green degree g . The bounds on $T_{1,2}$, $T_{2,1}$, $3M_2$ and $3M_3$ enter Φ with multipliers 6, 6, 2 and 2, so here

$$\Phi = W - 6\left\lfloor \frac{Cr}{2} \right\rfloor - 8\left\lfloor \frac{Cb}{2} \right\rfloor - 2\left\lfloor \frac{C_{33}g}{2} \right\rfloor.$$

The margins. The table gives, for each target:

- the caps;

- the number of admissible profiles;
- the *margin*, in this proof the least value of Φ , with the floors, over these profiles; and
- a profile attaining it.

target	n	(D_1, D_2, D_3)	mixed caps	profiles	margin	attained at
(3, 4, 7)	456	(57*, 128, 275)	(22, 154)	21	114	(57, 128, 270)
(3, 4, 8)	700	(78*, 178, 455*)	(27, 275)	91	312	(78, 178, 443)
(3, 5, 6)	683	(84*, 275, 345)	(39*, 87, 45, 154)	276	116	(62, 275, 345)

The mixed caps are (C, C_{33}) for $(3, 4, k)$, and $(C_{21}, C_{22}, C_{31}, C_{33})$ for $(3, 5, 6)$, that is, $R(4, 6) - 1$, $R(3, 3, 6) - 1$, $R(5, 5) - 1$ and $R(3, 4, 5) - 1$. A * marks a cap that rests on a starred input: $R(4, 7) \leq 58$, $R(4, 8) \leq 79$ and $R(5, 6) \leq 85$ in D_1 , and $R(4, 6) \leq 40$ in C_{21} . The cap $D_3 = 455$ for $(3, 4, 8)$ comes from the first bound of the proposition, and so it also rests on $R(4, 7) \leq 58$. The other caps come from unstarred entries of Table 3 and the theorems above.

The margins by hand. Every margin is positive, as step (iii) requires. The margins can be found by hand, as follows. Dropping the floors only lowers Φ , and the floor-free minima are already positive. For the weights $(2, 2, 2; 2, -1, -1)$ write $s = r + b$. Then

$$W = s(s-1) + g(g-1) - gs \quad \text{with} \quad g = n - 1 - s$$

$$3Cr + 4Cb = 3Cs + CD_2 - C(D_2 - b).$$

- (3, 4, 7): as $C = 22$ and $C_{33} = 154$ are even, the floors have no effect, and

$$\Phi = W - 66r - 88b - 154g = 3s^2 - 1277s + 133684 + 22(128 - b),$$

where $180 \leq s \leq 185$ and $b \leq 128$ by the caps. As $1277/6 > 185$, the quadratic in s decreases on this range, so the minimum is 114, at $(r, b, g) = (57, 128, 270)$.

- (3, 4, 8): with $C = 27$ and $C_{33} = 275$,

$$\Phi \geq W - 81r - 108b - 275g = 3s^2 - 1903s + 290871 + 27(178 - b),$$

where $244 \leq s \leq 256$ and $b \leq 178$. As $1903/6 > 256$, the minimum is 311, at $(78, 178, 443)$. There $27 \cdot 78$ and $27 \cdot 178$ are even and $275 \cdot 443$ is odd, so Φ itself, with the floors, is 312. Every other admissible profile has a larger floor-free value, so the margin is 312.

- (3, 5, 6): as in Theorem 6.3, the caps in the table give

$$\Phi = L - 6 \left\lfloor \frac{39b}{2} \right\rfloor - 4 \left\lfloor \frac{87b}{2} \right\rfloor - 6 \left\lfloor \frac{45g}{2} \right\rfloor - 4 \left\lfloor \frac{154g}{2} \right\rfloor \geq L - 291b - 443g,$$

with $291 = 6 \cdot \frac{39}{2} + 4 \cdot \frac{87}{2}$ and $443 = 6 \cdot \frac{45}{2} + 4 \cdot \frac{154}{2}$. Put $b = 275 - x$ and $g = 345 - y$. Then $r = 62 + x + y$, and $r \leq 84$ gives $x + y \leq 22$. At $(62, 275, 345)$, $L - 291b - 443g$ is 108 with gradient $(497, 179, 447)$, so the expansion after (5) with $\delta = (x + y, -x, -y)$ gives

$$L - 291b - 443g = 108 + 318x + 50y - 6xy.$$

For fixed $y \leq 22$ this increases in x , as $318 - 6y \geq 186$, so it is at least $108 + 50y \geq 108$, with equality only at $(62, 275, 345)$. There the floors add 3, 2 and 3, as $39 \cdot 275$, $87 \cdot 275$ and $45 \cdot 345$ are odd ($154 \cdot 345$ is even), so Φ itself is 116. Every other admissible profile has floor-free value at least 158, attained only at $(63, 275, 344)$, where $x = 0$ and $y = 1$, so the margin is 116.

Without the starred inputs. For the bounds without starred inputs, use instead:

- the published $R(4, 7) \leq 61$, $R(4, 8) \leq 84$ and $R(5, 6) \leq 87$ in D_1 ;
- $R(4, 6) \leq 41$ in C_{21} ; and
- for D_3 of $(3, 4, 8)$, $R(3, 4, 7) \leq 458$.

The reductions above then give the following.

- $(3, 4, 7)$, $n = 458$: the caps are now $(60, 128, 275)$ and $(22, 154)$, the floors have no effect, and

$$\Phi = 3s^2 - 1283s + 135198 + 22(128 - b)$$

on $182 \leq s \leq 188$. As $1283/6 > 188$, the minimum is 26, at $(60, 128, 269)$, and the margin is 26.

- $(3, 4, 8)$, $n = 705$: the caps are now $(83, 178, 457)$ and $(27, 275)$, and

$$\Phi \geq 3s^2 - 1918s + 296506 + 27(178 - b)$$

on $247 \leq s \leq 261$. As $1918/6 > 261$, the floor-free minimum is 271, at $(83, 178, 443)$. There the floors add 3, as $27 \cdot 83$ is odd, and 1, as $275 \cdot 443$ is odd, so $\Phi = 275$. Every other admissible profile has $s \leq 260$ and floor-free value at least 626, the value at $s = 260$ and $b = 178$, that is, at $(82, 178, 444)$, so the margin is 275.

- $(3, 5, 6)$, $n = 685$: the caps are now $(86, 275, 345)$ and $(40, 87, 45, 154)$, and

$$\Phi \geq L - 294b - 443g = 273 + 315x + 44y - 6xy,$$

with $b = 275 - x$, $g = 345 - y$, $r = 64 + x + y$ and $x + y \leq 22$. As $315 - 6y \geq 183$, the minimum is 273, only at $(64, 275, 345)$. There the floors add 2 and 3, as $87 \cdot 275$ and $45 \cdot 345$ are odd, so $\Phi = 278$. Every other admissible profile has floor-free value at least 317, attained only at $(65, 275, 344)$, so the margin is 278.

The computer check. A program (Section 10) also evaluates Φ , with the floors, at every admissible profile for both sets of inputs. It also tests its implementation of Lemma 6.1, for both weight vectors, on the 27 colourings of a triangle. This suffices, as each corner lies in exactly one triangle, so both sides of the identity are sums over triangles. $\square$

7 Four-vertex certificates for $R(4, 4, 4)$

Let c be a $(4, 4, 4; n)$ -colouring. By Lemma 2.1, with $R(3, 4, 4) \leq 77$, $R(2, 4, 4) = R(4, 4) = 18$ and $R(3, 3, 4) = 30$, every colour degree is at most 76, and for $u \in N_i(v)$ we have

$$|N_i(v) \cap N_i(u)| \leq 17 \quad \text{and} \quad |N_i(v) \cap N_j(u)| \leq 29 \quad \text{for } j \neq i.$$

Why triangles do not suffice. No colour is now required to be triangle-free, and the weighted triangle identities of Section 6 give nothing below the recurrence bound 230. Their analogue of Φ is now

$$W - \sum_i w_i \lfloor 17d_i/2 \rfloor - \sum_{i \neq j} (w_i + 2w_{ij}) \lfloor 29d_i/2 \rfloor,$$

with the sign conditions of Section 6 and also $w_1 \geq 0$, as M_1 need not vanish.

Under these conditions, which give rainbow triangles coefficient zero, no weights exclude any order $n \leq 229$ (for $n \geq 230$ no profile is admissible). Indeed, averaging the weights over the permutations of the colours preserves the conditions and, as the admissible profiles are closed under permuting entries, positivity of Φ at all of them. For symmetric weights, however, rainbow triangles have coefficient zero only if $w_{12} = w_{13} = w_{23} = 0$, and then

$$\Phi = w_1 \sum_i \left(\binom{d_i}{2} - \lfloor 17d_i/2 \rfloor - 2\lfloor 29d_i/2 \rfloor \right)$$

with $w_1 \geq 0$. Each summand is $d_i(d_i - 76)/2$, plus $\frac{3}{2}$ if d_i is odd, and so nonpositive for $0 \leq d_i \leq 76$.

A negative coefficient of X , which Section 6 also allows, does not exclude $n = 229$ either. There every profile is $(76, 76, 76)$, and $\Phi = 1368(w_{12} + w_{13} + w_{23})$ for every choice of weights. This is 1368 times the coefficient of X , so it is at most 0 whenever that coefficient is nonpositive. So we count patterns on four vertices instead.

- The probabilities that a random star is monochromatic or rainbow depend only on the degree profiles (Lemma 7.1).
- The caps bound the probabilities of certain four-vertex events by triangle densities (Lemma 7.2).
- Exact identities relate these quantities (Lemma 7.3).

Eliminating the triangle densities gives $\sum_v \Phi(\mathbf{d}(v)) \leq 0$ as in step (ii) (Theorem 7.4). Theorem 7.5 uses eight caps, three of which Theorem 7.4 also uses, and a sum-of-squares term. Notation introduced in this section is local to it.

The random quadruple. Choose an ordered quadruple (x_0, x_1, x_2, x_3) of distinct vertices uniformly at random and, independently, a uniformly random bijection σ from the colours 1, 2, 3 to labels R, B, G . Write

$$c_{ij} = \sigma(c(x_i x_j)),$$

and $N_R(v)$, $N_B(v)$, $N_G(v)$ for the neighbourhoods of v in the colours R, B, G of $\sigma \circ c$.

The law of (c_{ij}) is invariant under permutations of the indices $0, \dots, 3$, as the quadruple is uniform, and of the labels, as σ is uniform and independent of the quadruple. So labelled colourings of K_4 in one orbit of $S_4 \times S_3$ are equally likely. This holds for every c . No regularity is assumed.

We use the densities

$$\begin{aligned} s &= P(c_{01}=c_{02}=c_{03}=R), & t &= P(c_{01}=R, c_{02}=B, c_{03}=G), \\ m &= P(c_{01}=c_{02}=c_{12}=R), & b &= P(c_{01}=c_{02}=R, c_{12}=B). \end{aligned}$$

By label symmetry:

- $3s$ and $6t$ are the probabilities that the star x_0x_1, x_0x_2, x_0x_3 is monochromatic, respectively rainbow;

- $3m$ is the probability that the triangle $x_0x_1x_2$ is monochromatic.

Permuting x_0, x_1, x_2 and the labels takes the pattern $(c_{01}, c_{02}, c_{12}) = (R, R, B)$ to each of the 18 patterns with exactly two equal colours. So each has probability b , and $18b$ is the probability that the triangle has exactly two edges of the same colour.

Lemma 7.1 (Stars). *Let v be a vertex, $d_i = d_i(v)$ its colour degrees in c , and $(x)_3 = x(x-1)(x-2)$. Conditional on $x_0 = v$, the star x_0x_1, x_0x_2, x_0x_3 is monochromatic with probability $\sum_i (d_i)_3 / (n-1)_3$ and rainbow with probability $6d_1d_2d_3 / (n-1)_3$. As x_0 is uniform and the two events have unconditional probabilities $3s$ and $6t$, the star densities depend only on the degree profiles:*

$$s = \frac{1}{3n(n-1)_3} \sum_v \sum_i (d_i(v))_3, \quad t = \frac{1}{n(n-1)_3} \sum_v d_1(v)d_2(v)d_3(v).$$

Proof. Given $x_0 = v$, the triple (x_1, x_2, x_3) is uniform among the $(n-1)_3$ ordered triples of distinct vertices other than v , and neither event depends on σ . The star has all three edges of colour i for $(d_i)_3$ of these triples, and is rainbow for $6d_1d_2d_3$ of them. Here the factor 6 counts the ways to assign the three colours to x_1, x_2, x_3 . $\square$

Pattern probabilities. For a pattern $\tau = (c_{01}, c_{02}, c_{12})$ of the triangle $x_0x_1x_2$ and a condition κ on c_{03} and c_{13} , let $p(\tau; \kappa)$ be the probability that the triangle has pattern τ and κ holds, and $p(\tau)$ that of τ alone. Thus:

- $p(RRR) = m$;
- $p(\tau) = b$ for the 18 patterns τ with exactly two equal colours, among them RRB, RBR and RBB ;
- $p(RBG) = (1 - 3m - 18b)/6$, as the six rainbow patterns are equally likely.

The table lists the pairs (τ, κ) we use. In each, $c_{01} = R$, and κ says that x_3 lies in a set S :

- $S = N_R(x_0)$ if κ is $c_{03} = R$;
- $S = N_R(x_0) \cap N_\chi(x_1)$ if κ is $c_{03} = R, c_{13} = \chi$ for a label χ .

The column “cap” bounds $|S|$, and q is the cap less the number of vertices among x_1, x_2 that lie in S (Lemma 7.2).

name	$\tau = (c_{01}, c_{02}, c_{12})$	κ	cap	q	μ
A	RRR	$c_{03} = R$	76	74	
B'	RRR	$c_{03} = R, c_{13} = B$	29	29	
ℓ_1	RRR	$c_{03} = c_{13} = R$	17	16	12580
C, ℓ_2	RRB	$c_{03} = c_{13} = R$	17	17	70744
D, ℓ_3	RRB	$c_{03} = R, c_{13} = G$	29	29	45584
ℓ_4	RBR	$c_{03} = R$	76	75	40848
ℓ_5	RBR	$c_{03} = R, c_{13} = B$	29	29	40848
E, ℓ_6	RBB	$c_{03} = c_{13} = R$	17	17	96644
ℓ_7	RBG	$c_{03} = R, c_{13} = B$	29	29	40848
ℓ_8	RBG	$c_{03} = R, c_{13} = G$	29	29	142968

We write A, B', C, D and E for the probabilities $p(\tau; \kappa)$ of the rows so named. The prime avoids the label B . For a row marked ℓ_i , write $\tau_i, \kappa_i, q_i, \mu_i$ for its entries and

$$\ell_i = p(\tau_i; \kappa_i) - \frac{q_i}{222} p(\tau_i).$$

These are used at $n = 225$, where $n - 3 = 222$, in Theorem 7.5, and μ_i is the multiplier of ℓ_i in (7).

Lemma 7.2 (Pattern caps). *In a $(4, 4, 4; n)$ -colouring, with $n' = n - 3$, every row of the table satisfies $p(\tau; \kappa) \leq \frac{q}{n'} p(\tau)$. In particular,*

$$A \leq \frac{74}{n'} m, \quad B' \leq \frac{29}{n'} m, \quad C \leq \frac{17}{n'} b, \quad D \leq \frac{29}{n'} b, \quad E \leq \frac{17}{n'} b.$$

Proof. Condition on x_0, x_1, x_2 and σ , so that x_3 is uniform among the other n' vertices. If $x_0 x_1 x_2$ does not have pattern τ , the event has conditional probability 0. If it does, then $x_1 \in N_R(x_0)$, and Lemma 2.1 with $v = x_0$, and $u = x_1$ for the intersections, gives $|S| \leq 76, 17$ or 29 according as S is $N_R(x_0)$, $N_R(x_0) \cap N_R(x_1)$ or $N_R(x_0) \cap N_\chi(x_1)$ with $\chi \neq R$. These caps hold for the labels whatever σ is, as the three colours play symmetric roles.

Now:

- $x_0 \notin S$;
- $x_1 \in S$ only for $S = N_R(x_0)$, as $x_1 \notin N_\chi(x_1)$;
- $x_2 \in S$ exactly when κ holds with x_2 in place of x_3 , which given τ happens only in rows A and ℓ_1 .

So the conditional probability is

$$|S \setminus \{x_1, x_2\}|/n' \leq q/n'.$$

Averaging over x_0, x_1, x_2 and σ gives the lemma, and the five inequalities follow as $p(RRR) = m$ and $p(RRB) = p(RBB) = b$. $\square$

Lemma 7.3 (Three identities). *For every 3-colouring of K_n with $n \geq 4$,*

$$(I) \quad 9m + 18b = 1 + 6s - 6t, \quad (II) \quad 3A + 6B' + 6D + 6E = 2s + m, \quad (III) \quad B' = C.$$

Proof. Identity (III). Exchanging x_2 and x_3 preserves the law of (c_{ij}) . It maps the event of row C , $c_{01} = c_{02} = c_{03} = c_{13} = R$ and $c_{12} = B$, onto that of row B' , $c_{01} = c_{02} = c_{12} = c_{03} = R$ and $c_{13} = B$.

Identity (I). Both sides equal $3P(c_{01}=c_{02})$. By symmetry in x_0, x_1, x_2 , this is the expected number of corners of the triangle $x_0x_1x_2$ whose two edges have the same colour:

- 3 if the triangle is monochromatic (probability $3m$);
- 1 if it has exactly two edges of the same colour ($18b$);
- 0 otherwise.

By symmetry in x_1, x_2, x_3 , it is also the expected number of pairs of edges of the star at x_0 with the same colour:

- 3 if the star is monochromatic ($3s$);
- 0 if it is rainbow ($6t$);
- 1 otherwise ($1 - 3s - 6t$).

So

$$9m + 18b = 9s + (1 - 3s - 6t) = 1 + 6s - 6t.$$

Identity (II): reduction. Split s by c_{12} and m by c_{03} . Exchanging the labels B and G preserves the law, so the cases $c_{12} = G$ of s and $c_{03} = G$ of m have the same probabilities as $c_{12} = B$ and $c_{03} = B$. The case $c_{12} = R$ of s and the case $c_{03} = R$ of m are both the event of row A . Splitting further by c_{13} :

- the case $c_{12} = B$ of s gives C , D and Y_s for $c_{13} = R, G, B$;
- the case $c_{03} = B$ of m gives B' , E and Y_m for $c_{13} = R, B, G$, the values B' and E after exchanging x_0 with x_1 , respectively x_2 with x_3 .

Here

$$Y_s = P(c_{01}=c_{02}=c_{03}=R, c_{12}=c_{13}=B), \quad Y_m = P(c_{01}=c_{02}=c_{12}=R, c_{03}=B, c_{13}=G).$$

So

$$s = A + 2(C + D + Y_s) \quad \text{and} \quad m = A + 2(B' + E + Y_m),$$

and with (III),

$$2s + m = 3A + 6B' + 4D + 2E + 4Y_s + 2Y_m.$$

Hence (II) is equivalent to

$$2Y_s + Y_m = D + 2E.$$

Identity (II): the twelve terms. To prove this, split each of Y_s , Y_m , D and E further by c_{23} . Each of the twelve resulting terms is the probability of one labelled colouring of K_4 , and terms whose colourings lie in one $S_4 \times S_3$ -orbit are equal. The table gives, for each term, the classes of colours R , B and G , in this order, as graphs on $\{x_0, \dots, x_3\}$. Here the paw is a triangle with a pendant edge, P_3 the path with two edges, and $\emptyset$ an empty class.

	$c_{23} = R$	$c_{23} = B$	$c_{23} = G$
Y_s	paw, $P_3, \emptyset$ (π_1)	$K_{1,3}, K_3, \emptyset$ (π_2)	$K_{1,3}, P_3, K_2$ (π_3)
Y_m	paw, K_2, K_2 (π_4)	K_3, P_3, K_2 (π_5)	K_3, K_2, P_3 (π_5)
D	paw, K_2, K_2 (π_4)	$K_{1,3}, P_3, K_2$ (π_3)	$K_{1,3}, K_2, P_3$ (π_3)
E	paw, $P_3, \emptyset$ (π_1)	$K_3, K_{1,3}, \emptyset$ (π_2)	K_3, P_3, K_2 (π_5)

Identity (II): the orbits. Colourings with the same multiset of classes lie in one orbit. Indeed, each multiset contains a paw, $K_{1,3}$ or K_3 . Permuting labels and vertices, we may assume that both colourings have this class, in the same colour. Its stabiliser in S_4 , together with the exchange of the other two labels, acts transitively on the splittings of its complement into the remaining two classes. Writing $\pi_1, \dots, \pi_5$ for the five common values, as marked, the rows give

$$Y_s = \pi_1 + \pi_2 + \pi_3, \quad Y_m = \pi_4 + 2\pi_5, \quad D = \pi_4 + 2\pi_3, \quad E = \pi_1 + \pi_2 + \pi_5,$$

and hence

$$2Y_s + Y_m = 2\pi_1 + 2\pi_2 + 2\pi_3 + \pi_4 + 2\pi_5 = D + 2E.$$

□

Theorem 7.4. $R(4, 4, 4) \leq 226$.

Proof. Step 1: combining the caps. Suppose that a $(4, 4, 4; 226)$ -colouring exists, so that $n' = 223$. For $0 \leq \lambda \leq 1$, add the five inequalities of Lemma 7.2 with multipliers $\frac{1}{2}$, λ , $1 - \lambda$, 1 and 1. The multipliers λ and $1 - \lambda$ divide the one probability $B' = C$ between its two caps. By (III) and (II), the left-hand side is $\frac{1}{2}A + B' + D + E = \frac{1}{6}(2s + m)$, so

$$\frac{1}{6}(2s + m) \leq \frac{(37 + 29\lambda)m + (63 - 17\lambda)b}{n'}.$$

Step 2: eliminating m . Substituting $b = (1 + 6s - 6t)/18 - m/2$ from (I) gives m the coefficient $(11 + 75\lambda)/(2n')$ on the right. If this is at most $\frac{1}{6}$, the net term in m on the left is nonnegative, as $m \geq 0$, and can be dropped, leaving an inequality in s and t . As $1 + 6s - 6t = 9m + 18b \geq 0$, this is strongest when the coefficient $63 - 17\lambda$ of b is least. So we take the largest such λ , namely $\lambda = (n' - 33)/225$, here $\frac{38}{45}$, for which the terms in m cancel.

Step 3: the star inequality. Multiplying by $18n'$ then gives $6n's \leq (63 - 17\lambda)(1 + 6s - 6t)$, and substituting $\lambda = (n' - 33)/225$ and multiplying by 225 gives

$$1350n's \leq (14736 - 17n')(1 + 6s - 6t).$$

For $n' = 223$, dividing by 5 gives $60210s \leq 2189(1 + 6s - 6t)$, that is,

$$15692 \cdot 3s + 2189 \cdot 6t \leq 2189, \tag{6}$$

or $15692P(\text{monochromatic star}) + 2189P(\text{rainbow star}) \leq 2189$ for the star at x_0 .

Step 4: the profiles. Every vertex has a profile with entries at most 76 summing to 225. Up to order, these are $(73, 76, 76)$, $(74, 75, 76)$ and $(75, 75, 75)$. Given $x_0 = v$, Lemma 7.1 gives

the conditional probabilities $\sum_i (d_i)_3 / (n-1)_3$ and $6d_1d_2d_3 / (n-1)_3$ of a monochromatic and a rainbow star, where $d_i = d_i(v)$. With these in place of $3s$ and $6t$, the left side of (6) minus 2189 becomes

$$\Phi(\mathbf{d}(v)) = \frac{15692 \sum_i (d_i)_3 + 6 \cdot 2189 d_1 d_2 d_3 - 2189 (n-1)_3}{(n-1)_3},$$

where

$$(n-1)_3 = 225 \cdot 224 \cdot 223 = 11,239,200 \quad \text{and} \quad 2189 (n-1)_3 = 24,602,608,800.$$

Step 5: the contradiction. As x_0 is uniform, (6) says that

$$\sum_v \Phi(\mathbf{d}(v)) \leq 0.$$

But Φ is symmetric in the d_i and positive at every profile:

profile	$\sum_i (d_i)_3$	$d_1 d_2 d_3$	numerator	$\Phi(\mathbf{d})$
(73, 76, 76)	1,216,776	421,648	28,965,024	451/175
(74, 75, 76)	1,215,894	421,800	17,121,048	457/300
(75, 75, 75)	1,215,450	421,875	11,138,850	111/112

This is a contradiction. □

Where the multipliers come from. The multipliers were found by linear programming at $n = 228$, where every profile is (75, 76, 76), which selected the five caps used here. With these caps, (II), (III) and the cancellation of m determine the multipliers up to a common factor. At $n = 226$ exact linear algebra gives $\frac{1}{2}$, $\frac{38}{45}$, $\frac{7}{45}$, 1 and 1. The proof does not depend on this.

Why it stops here. At $n = 225$, where $n' = 222$ and $\lambda = \frac{21}{25}$, the inequality $1350 n' s \leq (14736 - 17n')(1 + 6s - 6t)$ of the proof, divided by 54, becomes $5550 s \leq 203(1 + 6s - 6t)$, that is,

$$1444 \cdot 3s + 203 \cdot 6t \leq 203$$

in place of (6).

The profiles now have entries at most 76 summing to 224. Up to order they are (72, 76, 76), (73, 75, 76), (74, 74, 76) and (74, 75, 75). At each, the left side minus 203, computed as in the proof, is negative. In this order the values are $-1369/1561$, $-1597/1561$, $-239/223$ and $-28009/24976$, the largest being $-1369/1561 \approx -0.88$. Since the proof takes the best λ for which the term in m can be dropped, no choice of λ lets this argument exclude 225.

A sum of squares. Theorem 7.5 combines the eight inequalities $\ell_i \leq 0$ of the table, which hold in a (4, 4, 4; 225)-colouring by Lemma 7.2, and a sum-of-squares term $Q \geq 0$, through one linear identity checked by computer. Three of the ℓ_i are the caps on C , D and E . The caps on A and B' are not used.

For a 3-colouring of K_{225} , define Q as follows. Identify the labels R, B, G with the residues $0, -1, 1$ modulo 3, and let $a(\alpha, \beta) \in \{-1, 0, 1\}$ be congruent to $\alpha + \beta$. For each vertex $w \neq x_0, x_1$, let

$$a_w = a(\sigma(c(x_0w)), \sigma(c(x_1w))) \quad \text{and} \quad Z = \sum_w a_w,$$

so that Z is a sum of $n - 2 = 223$ terms depending on x_0, x_1 and σ . Then

$$Q := \mathbb{E}[\mathbf{1}_{c_{01}=R} (Z/223)^2] = \frac{222}{223} \mathbb{E}[\mathbf{1}_{c_{01}=R} a_{x_2} a_{x_3}] + \frac{1}{223} \mathbb{E}[\mathbf{1}_{c_{01}=R} a_{x_2}^2] \geq 0.$$

Indeed,

$$Z^2 = \sum_w a_w^2 + \sum_{w \neq w'} a_w a_{w'},$$

where the second sum has $223 \cdot 222$ terms. Given x_0, x_1 and σ , the indicator is fixed and (x_2, x_3) is a uniform ordered pair of distinct vertices among the other 223. So the conditional means of $a_{x_2}^2$ and $a_{x_2} a_{x_3}$ are the averages of the terms of the two sums. As $a_{x_j} = a(c_{0j}, c_{1j})$, both expectations on the right are combinations of probabilities of colour patterns on $x_0, \dots, x_3$.

Any real function a would give $Q \geq 0$. This one is needed only for the identity (7) below.

Theorem 7.5 (identity checked by computer). $R(4, 4, 4) \leq 225$.

Proof. Step 1: the identity. Besides s and t we use a third star density,

$$\eta = P(c_{01}=c_{02}=R, c_{03}=B).$$

As for b , each of the 18 patterns (c_{01}, c_{02}, c_{03}) of the star with exactly two equal colours has probability η , so $3s + 18\eta + 6t = 1$. Thus s and t determine η . We keep all three so that the left side of (7) has no constant term. For every 3-colouring of K_{225} ,

$$\frac{35020}{3} s + \frac{39040}{3} \eta - 24012 t = \sum_{i=1}^8 \mu_i \ell_i - 5129 Q. \quad (7)$$

Step 2: checking the identity. With Q expanded as above, both sides are linear combinations of the probabilities of the $3^6 = 729$ labelled colourings of K_4 on $(x_0, \dots, x_3)$. By the invariance noted at the start of the section, these probabilities are constant on each of the 15 orbits of $S_4 \times S_3$. So it suffices to check (7) for the uniform law on each orbit. We did this in exact rational arithmetic, with a program (`verify_r444.py`, Section 10) that shares no code with the search that found the multipliers.

Step 3: the sign of the right side. In a $(4, 4, 4; 225)$ -colouring the right side of (7) is at most 0, as each μ_i is positive, each $\ell_i \leq 0$ and $Q \geq 0$.

Step 4: the left side at a vertex. Given $x_0 = v$, write $d_i = d_i(v)$. By Lemma 7.1 and label symmetry, the events defining s and t have conditional probabilities $\sum_i (d_i)_3 / (3(n-1)_3)$ and $d_1 d_2 d_3 / (n-1)_3$, where $(n-1)_3 = 224 \cdot 223 \cdot 222$.

For η , the colours of c that σ maps to R and B form a uniformly random ordered pair (i, j) of distinct colours. Given it, $x_0 x_1$ and $x_0 x_2$ have colour i and $x_0 x_3$ colour j for $d_i(d_i - 1)d_j$

of the $(n-1)_3$ triples (x_1, x_2, x_3) . So the event defining η has conditional probability $\sum_{i \neq j} d_i(d_i-1)d_j/(6(n-1)_3)$. Multiplying the conditional value of the left side of (7) by $18(n-1)_3$ and using $\sum_{j \neq i} d_j = n-1-d_i = 224-d_i$ gives

$$J(\mathbf{d}) = 70040 \sum_i (d_i)_3 + 39040 \sum_i d_i(d_i-1)(224-d_i) - 432216 d_1 d_2 d_3.$$

Step 5: the contradiction. As x_0 is uniform, the left side of (7) is the average over v of $J(\mathbf{d}(v))/(18(n-1)_3)$. So $\sum_v J(\mathbf{d}(v)) \leq 0$, and J plays the role of Φ . But J is symmetric, and positive at each of the four profiles listed after Theorem 7.4:

profile	$\sum_i (d_i)_3$	$\sum_i d_i(d_i-1)(224-d_i)$	$d_1 d_2 d_3$	$J(\mathbf{d})$
(72, 76, 76)	1,201,440	2,464,224	415,872	605,630,208
(73, 75, 76)	1,200,126	2,464,206	416,100	414,349,680
(74, 74, 76)	1,199,688	2,464,200	416,176	350,589,504
(74, 75, 75)	1,199,244	2,464,200	416,250	287,507,760

This is a contradiction. □

How the certificate was found. The function a , the μ_i and 5129 were found numerically. The proof does not depend on how. At $n = 225$, consider the linear programme in the probabilities of the 14 orbits of colourings of K_4 that are not monochromatic, subject to the following constraints:

- the star densities are a mixture of those of the admissible profiles;
- the degree bounds $72 \leq d_i \leq 76$ and the caps 17 and 29 of Lemma 2.1 hold, conditioned on each of the 27 patterns of $x_0 x_1 x_2$.

This programme is feasible.

At a feasible point, the 10×10 matrix of the quadratic form $(v_0, v) \mapsto \mathbb{E}[\mathbf{1}_{c_{01}=R}(v_0 + Z_v/223)^2]$, where Z_v is Z with a replaced by a function v on pairs of labels, has negative eigenvalues. For an eigenvector of the most negative one, the signs of its nine entries indexed by pairs of labels give a . The single constraint $Q \geq 0$ then makes the programme infeasible, and the μ_i and 5129 are a dual solution, rationalised and then checked exactly. Given the terms of (7), the identity determines its coefficients up to a common factor, so $\mu_4 = \mu_5 = \mu_7$ is forced.

The residues modulo 3 in the definition of Q only encode the nine values of a . They are unrelated to the argument of [5].

Other values of n . With the following changes, the same method gives exact certificates for $219 \leq n \leq 224$ (Section 10):

- several squares $\mathbf{1}_{c_{01}=R}(v_0 + Z_v/(n-2))^2$ in place of Q ;
- three to five caps of the kind in Lemma 7.2;

- in place of the identity, conditions (i) and (ii) of Lemma 8.1 on four vertices, for the star term $c_s \mathbf{1}_s + c_\eta \mathbf{1}_\eta + c_t \mathbf{1}_t$, where $\mathbf{1}_s$, $\mathbf{1}_\eta$ and $\mathbf{1}_t$ are the indicators of the events defining s , η and t , these caps and these squares; condition (i) is checked on the 14 orbits other than that of a monochromatic K_4 , which has probability 0.

Theorem 8.2 supersedes them.

8 Five-vertex certificates with local caps

Section 7 averages over a random ordered quadruple of vertices. The same idea with a random ordered 5-tuple, the caps of Lemma 2.3 and many squares gives most of our strongest bounds.

The certificates are too large to print. Instead we describe their form, prove that a certificate of this form excludes the order n , and give their sizes and margins. Every certificate was found in floating point, by semidefinite programming, and then rounded to rational numbers. The solvers were:

- for Theorem 8.2, a dual log-barrier interior-point method written for this work;
- for Theorem 8.3 and the mixture-model certificate for $R(4, 4, 4) \leq 200$ (see after Theorem 8.2), Clarabel [12], with the star and row multipliers re-solved by HiGHS [15] after the squares are rounded;
- for $R(3, 3, k)$, a cutting-plane method for the semidefinite constraints, with the linear programmes solved by HiGHS.

Every certificate is checked in exact rational arithmetic by a program that shares no code with the search (Section 10).

8.1 Certificates

Let c be a $(\mathbf{t}; n)$ -colouring and $s \in \{4, 5\}$. Let Γ be the group of permutations γ of the colours with $t_{\gamma(i)} = t_i$ for all i . For $\mathbf{t} = (3, 3, k)$ we use instead the group of order 2 exchanging red and blue, also when $k = 3$; everything below holds for any group of colour permutations fixing $\mathbf{t}$. Choose an ordered s -tuple $(x_0, \dots, x_{s-1})$ of distinct vertices uniformly at random and, independently, a uniform $\sigma \in \Gamma$. Let P be the colouring $x_i x_j \mapsto \sigma(c(x_i x_j))$ of K_s . Neighbourhoods below are taken in $\sigma \circ c$, which is again $\mathbf{t}$ -avoiding.

As in Section 7, the law of P is invariant under $S_s \times \Gamma$. So

$$\mathbb{E}f(P) = \mathbb{E}(\text{Sym}f)(P)$$

for every function f on colourings of K_s , where $\text{Sym}f$ averages f over the orbit; and P has no colour- i K_{t_i} for any i .

We use three kinds of terms.

- Stars.* For a pattern π of colours on the star $x_0 x_1, \dots, x_0 x_{s-1}$, the probability that the star has pattern π , given $x_0 = v$, is a polynomial of degree $s - 1$ in $\mathbf{d}(v)$, as in Lemma 7.1. So for $G = \sum_{\pi} y_{\pi} \mathbf{1}[\pi]$ we have

$$\mathbb{E}G = \mathbb{E}_v J_G(\mathbf{d}(v)),$$

where v is a uniform vertex and J_G is an explicit polynomial.

- (b) *Rows*. Fix a colouring ρ of the pairs among $x_0, \dots, x_{s-2}$, a nonempty set Y of these vertices and colours $a(y)$, $y \in Y$, and let

$$S = \bigcap_{y \in Y} N_{a(y)}(y).$$

Lemma 2.3, with $\mathbf{w}$ read off from ρ , caps $|S|$. Let q be this cap less the number of vertices among $x_0, \dots, x_{s-2}$ that lie in S , which ρ determines.

- Given $x_0, \dots, x_{s-2}$, the vertex x_{s-1} is uniform among the other $n - s + 1$. So the *cap row*

$$g = \mathbf{1}[\rho](\mathbf{1}[x_{s-1} \in S] - q/(n - s + 1))$$

has $\mathbb{E} g \leq 0$.

- Let $Y = \{y\}$ and $a(y) = i$, so that $S = N_i(y)$, and let q be the lower bound

$$d_i(y) \geq n - 1 - \sum_{l \neq i} D_l$$

less the number of vertices among $x_0, \dots, x_{s-2}$ that lie in S . Then the *lower row*

$$g = \mathbf{1}[\rho](q/(n - s + 1) - \mathbf{1}[x_{s-1} \in S])$$

has $\mathbb{E} g \leq 0$.

- For $\mathbf{t} = (3, 3, k)$ we also use *profile rows*: functions g of the colours of x_0x_1 , x_0x_2 and x_1x_2 with $\mathbb{E} g \leq \mathbb{E}_v \beta(\mathbf{d}(v))$ for an explicit function β . For fixed σ , given $x_0 = v$, the probability that x_0x_1 and x_0x_2 have colour i and x_1x_2 colour j is $2e_j(N_i(v))/((n-1)(n-2))$, with N_i taken in $\sigma \circ c$. So bounds on $e_j(N_i(v))$ give three kinds of profile row. Each β below is the bound for fixed σ , with d_i the colour- i degree of v in $\sigma \circ c$. The row uses its average over $\sigma \in \Gamma$, that is, over the exchange of red and blue, as a function of $\mathbf{d}(v)$:

- the *upper floor row*, the indicator g of this event, with $\beta(\mathbf{d}) = 2[C_{ij}d_i/2]/((n-1)(n-2))$, by Corollary 2.2;
- the *lower floor row* $-g$, with $\beta(\mathbf{d}) = -2[d_i \max(0, d_i - 1 - \sum_{j' \neq j} C_{ij'})/2]/((n-1)(n-2))$, as each $u \in N_i(v)$ has at least $d_i - 1 - \sum_{j' \neq j} C_{ij'}$ colour- j neighbours in $N_i(v)$;
- the *recursive row*, minus the indicator that x_0x_1 and x_0x_2 are green and x_1x_2 is red or blue, with $\beta(\mathbf{d}) = -2m_{k-1}(d_3)/((n-1)(n-2))$, by the valid lower bound $e_H(N_3(v)) \geq m_{k-1}(d_3(v))$ from the proof of Lemma 3.3.

For cap and lower rows put $\beta = 0$.

- (c) *Squares*. Fix a root, a vertex, an edge or a triangle with given colours, placed on the first one, two or three vertices of the tuple. Give every vertex w of K_n outside the root (for a vertex root, every pair of such vertices, ordered or unordered) a *type*, the colours of its edges to the root (and between the pair). For a constant θ_0 and a vector θ indexed by the types, let

$$Z = \sum_w \theta_{\text{type}(w)},$$

summed over these vertices (or pairs), and let N be the number of terms. Then

$$\mathbf{1}[\text{root}] (\theta_0 + Z/N)^2 \geq 0.$$

As in the definition of Q in Section 7, averaging over the remaining vertices of the tuple turns its expectation into that of a function of P , provided the positions of the tuple outside the root can hold two summed vertices (or two disjoint pairs). Indeed, the square expands into θ_0^2 , $2\theta_0$ times a mean over one vertex (or pair) outside the root, and Z^2/N^2 . The last splits by how the two summed vertices or pairs overlap: equal or distinct vertices, and pairs sharing two vertices, one or none. Each part is a mean over at most four vertices outside the root, and so it is the expectation of a function of P . In Lemma 8.1, Q_j denotes this function of P , which depends on n through N . The pairs are ordered in the certificates for $R(3, 3, k)$ and those of Theorem 8.3, and unordered in those of Theorem 8.2. In the certificates of Theorem 8.3 a square at a vertex root may also combine the means over single vertices and over pairs (that is, Z/N is replaced by $Z_1/N_1 + Z_2/N_2$, with Z_1 summed over single vertices and Z_2 over ordered pairs). For $\mathbf{t} = (4, 4, 4)$ we also use *localised* squares at a red edge x_0x_1 :

- $\mathbf{1}[\text{root}] (q - |S|)(\theta_0 + Z/N)^2 \geq 0$, where S is $N_i(x_0) \cap N_j(x_1)$, $N_i(x_0)$ or $N_j(x_1)$ with x_0, x_1 removed, and q is the cap of Lemma 2.3 less the number of removed vertices that lie in the set; and
- $\mathbf{1}[\text{root}] (|S| - q)(\theta_0 + Z/N)^2 \geq 0$ for $S = N_i(x_0) \setminus \{x_1\}$, with q the lower degree bound less $\mathbf{1}[x_1 \in N_i(x_0)]$.

Writing $|S| = \sum_u \mathbf{1}[u \in S]$ over the vertices $u \neq x_0, x_1$, and splitting again by whether u equals one of the two summed vertices, turns the expectation into that of a function of P on at most three vertices outside the root.

Lemma 8.1. *Suppose that for a star term G , rows g_r with $\mathbb{E} g_r \leq \mathbb{E}_v \beta_r(\mathbf{d}(v))$, squares Q_j (so $\mathbb{E} Q_j \geq 0$) and multipliers $\mu_r, \lambda_j \geq 0$,*

(i) *the inequality*

$$\text{Sym}\left(G - \sum_r \mu_r g_r + \sum_j \lambda_j Q_j\right) \leq 0$$

holds at every colouring of K_s that has no colour- i K_{t_i} for any i , and

(ii) *the inequality*

$$J_G(\mathbf{d}) - \sum_r \mu_r \beta_r(\mathbf{d}) > 0$$

holds at every admissible profile $\mathbf{d}$.

Then no $(\mathbf{t}; n)$ -colouring exists. If instead of (ii),

$$(n-1)\left(J_G(\mathbf{d}) - \sum_r \mu_r \beta_r(\mathbf{d})\right) + d_1 + d_2 \geq h^*$$

at every admissible profile, and $\mathbf{t} = (3, 3, k)$, then $\lceil nh^*/2 \rceil$ is a valid lower bound $m_k(n)$.

Proof. By (i) and the invariance of the law of P ,

$$0 \geq \mathbb{E}\left(G - \sum_r \mu_r g_r + \sum_j \lambda_j Q_j\right) \geq \mathbb{E}_v\left(J_G(\mathbf{d}(v)) - \sum_r \mu_r \beta_r(\mathbf{d}(v))\right),$$

which (ii) makes positive, as every profile is admissible. Under the alternative hypothesis the same chain gives

$$0 \geq (h^* - \mathbb{E}_v(d_1(v) + d_2(v)))/(n-1),$$

so the average H -degree is at least h^* , and $e(H) \geq nh^*/2$. $\square$

The star term absorbs any constant, as $\sum_{\pi} \mathbf{1}[\pi] = 1$. So (i) and (ii) say that the profile polynomial, corrected by the rows, is positive, while the combination of caps and squares makes the whole function nonpositive.

Theorems 7.4 and 7.5 are the case $s = 4$, with no square or one, and with identities in place of the inequality (i). The certificates for $219 \leq n \leq 224$ described at the end of Section 7 are the case $s = 4$ with several squares. In the floating-point search described at the start of the section, the vectors of the squares are rounded to integers and the linear part is re-solved before the multipliers are rationalised.

Checking (i) needs only the orbits of $S_s \times \Gamma$ on colourings of K_s without a forbidden clique. For $s = 5$ there are 133 for $(4, 4, 4)$, 193 for $(3, 3, k)$ with $k \geq 6$, 273 for $(3, 4, 4)$, and between 287 and 566 for the other targets. Checking (ii) needs every admissible profile.

8.2 Results

Theorem 8.2 (certificate checked by computer). $R(4, 4, 4) \leq 198$.

Proof. A certificate as in Lemma 8.1 for $\mathbf{t} = (4, 4, 4)$, $n = 198$ and $s = 5$.

- Its star term has four classes, the multiplicity patterns 4, 3+1, 2+2 and 2+1+1 of the colours on $x_0x_1, \dots, x_0x_4$, so J_G is a quartic in the profile.
- It has 1581 cap and lower rows, conditioned on the colouring of $x_0, \dots, x_3$, with Y of size 1 to 4.
- It has 233 squares:
 - 113 ordinary squares: at a red edge (9 types), at a monochromatic, a two-coloured or a rainbow triangle (27 types), and at a vertex (18 types of unordered pairs); and
 - 120 localised squares at a red edge, with the constants $q = 17, 29, 75$ and 76 (the cap 76 less 1 gives 75 for $N_1(x_0)$, which contains x_1), and with the lower degree bound.
- The inputs are $R(3, 4, 4) \leq 77$, so that every colour degree is at most 76, $R(3, 3, 4) = 30$, $R(3, 3, 3) = 17$, $R(4, 4) = 18$, $R(3, 4) = 9$ and $R(3, 3) = 6$.
- The multipliers are dyadic rationals and the square vectors are integral.

In exact arithmetic, the largest value in (i) over the 133 orbits is $-3.05 \cdot 10^{-9}$, and the least value in (ii) over the 528 admissible profiles is $3.05 \cdot 10^{-9}$, so its margin, as defined below, is about $6.1 \cdot 10^{-9}$. A second program views every term of the certificate as a function of the colouring of K_6 on $(x_0, \dots, x_5)$, which changes no expectation, as no term involves x_5 . It evaluates (i) over the 3929 orbits of $S_6 \times S_3$ on colourings of K_6 without a monochromatic K_4 , by exact integer counts over all 3^{15} labelled colourings, and finds the largest value in (i) to be $-3.05 \cdot 10^{-9}$ as well (to these digits; exactly, it is slightly smaller) and the same least value in (ii). $\square$

Comparison with other versions. Theorem 8.2 supersedes Theorems 7.4 and 7.5. Below, the relaxation at order n is *decided* if the floating-point optimum of its semidefinite programme is positive beyond solver precision, so that a certificate is to be expected, and *undecided* if this optimum is 0 to solver precision, as in Remark 8.6. Here the programme maximises the margin of a certificate of the given form under a normalisation, the *margin* being the least slack in (i) of Lemma 8.1 over the colourings of K_s plus the least slack in (ii) over the admissible profiles. As the margin scales with the certificate, a certificate of this form exists exactly when this optimum is positive. So an optimum of 0 to solver precision, or a slightly negative one as in Remark 8.6, means that, up to solver precision, none exists.

- Without the localised squares, the same method gives certificates for $n = 218, 212, 208$ and 204, of which only the last is in the ancillary files, and in floating point its relaxation is decided down to $n = 200$.
- The five-vertex model of Theorem 8.3 (the *mixture model*, described under “Certificates and margins” below) also gives $R(4, 4, 4) \leq 200$. It differs from the model of Theorem 8.2 in its terms: a coefficient for each star pattern, squares over ordered pairs that may combine single vertices and pairs, and no localised squares. Its certificate, `flag5_r34k/certificates/R444_200.json`, is in the ancillary files.
- The four-vertex method of Section 7 with several squares reaches 219, and its full semidefinite relaxation is undecided at 218.

Why it stops here. In the certificate of Theorem 8.2 the multiplier weight sits on rows with small caps: the cap rows

$$|N_i(u) \cap N_i(v)| \leq 17$$

for an edge uv of colour i , which come from $R(4, 4) = 18$, and the four-rooted cap rows

$$|N_i(x) \cap N_i(y) \cap N_j(z) \cap N_j(w)| \leq 3$$

for an edge xy of colour i and an edge zw of colour $j \neq i$, which come from $R(2, 2, 4) = 4$. The degree cap 76 carries almost no weight. Correspondingly, a better bound for $R(3, 4, 4)$ alone did not help in our runs. With the hypothesis $R(3, 4, 4) \leq 75$ in place of 77 the relaxation at $n = 197$ is still undecided (we did not rerun this model with our bound $R(3, 4, 4) \leq 73$ of Theorem 8.3), and the mixture model stops at 200 with $R(3, 4, 4) \leq 73$ as with 77.

At $n = 197$ the relaxation is undecided to solver precision, and it stays so with either of two six-vertex extensions: localised squares at a triangle root, or squares at an edge root summed over pairs of vertices outside the root, as at a vertex root. Even under the extra hypothesis that every vertex has a regular profile, with entries differing by at most 1, the relaxation is decided numerically at 197 but not at 196. So conditioning on profiles could gain at most one order.

Certificates and margins. For the certificates of Theorem 8.3, and the mixture-model certificate for $R(4, 4, 4) \leq 200$ above, the star term has a coefficient for each of the 3^4 colour patterns of the star. The search treats the law of the profile of x_0 as an unknown mixture of the admissible profiles; as for every certificate of Lemma 8.1, the check is then made at each admissible profile. Each certificate is a function F , the star term plus nonnegative multiples of cap and lower rows minus nonnegative multiples of squares, with

$$\min \text{Sym } F > \max J_G,$$

the minimum over the colourings of K_5 without a forbidden clique and the maximum over the admissible profiles. For any constant C between these two values, $C - F$, with C absorbed into its star term, satisfies (i) and (ii) of Lemma 8.1. For each such C , the least slack in (i) plus the least slack in (ii) is the same, so the margin, as defined above, is

$$\min \text{Sym } F - \max J_G.$$

Theorem 8.3 (certificates checked by computer).

$$\begin{aligned} R(3, 4, 4) &\leq 73, & R(3, 4, 5) &\leq 137, & R(3, 5, 5) &\leq 316, \\ R(3, 4, 6) &\leq 247, & R(3, 4, 7) &\leq 409, \\ R(3, 5, 6) &\leq 622 & \text{and} & R(3, 4, 8) &\leq 626. \end{aligned}$$

With the starred inputs of Table 3,

$$\begin{aligned} R(3, 4, 6) &\leq 245, & R(3, 4, 7) &\leq 406, \\ R(3, 5, 6) &\leq 619 & \text{and} & R(3, 4, 8) &\leq 625. \end{aligned}$$

Proof. One certificate for each bound, with $s = 5$, checked in exact arithmetic over all 3^{10} labelled colourings of K_5 :

target	n	orbits	profiles	margin	new bounds used
(3, 4, 4)	73	273	10	$2.2 \cdot 10^{-5}$	none
(3, 4, 5)	137	551	231	$5.4 \cdot 10^{-7}$	none
(3, 5, 5)	316	287	6	$1.5 \cdot 10^{-7}$	73, 137
(3, 4, 6)	247	552	171	$4.1 \cdot 10^{-7}$	73, 137
(3, 4, 7)	409	552	378	$5.5 \cdot 10^{-8}$	73, 137, 247
(3, 5, 6)	622	566	378	$3.6 \cdot 10^{-8}$	73, 137, 247, 316
(3, 4, 8)	626	552	1128	$2.7 \cdot 10^{-8}$	73, 137, 247, 409

The table rows are for the bounds with published inputs.

- $R(3, 4, 4) \leq 73$ uses only $R(3, 3) = 6$, $R(3, 4) = 9$, $R(4, 4) = 18$ and $R(3, 3, 4) = 30$, and needs the exchange of blue and green, which lies in Γ .
- $R(3, 4, 5) \leq 137$ uses in addition $R(3, 5) = 14$, $R(4, 5) = 25$, $R(3, 3, 3) = 17$, $R(3, 3, 5) \leq 57$ and $R(3, 4, 4) \leq 77$.
- The others also use
 - the values $R(3, k)$ with $k \leq 8$,
 - the published bounds $R(5, 5) \leq 46$, $R(4, 6) \leq 41$, $R(4, 7) \leq 61$, $R(4, 8) \leq 84$ and $R(5, 6) \leq 87$,
 - the bounds of the theorem listed in the table, and
 - where they enter a cap, the hand bounds of Theorems 4.1, 4.3 and 4.5.

With $R(3, 4, 4) \leq 77$ in place of 73, the (3, 5, 5) certificate remains valid, with the same margin, once its one cap constant that involves $R(3, 4, 4)$ is recomputed, from 72 to 76. The

stored certificate has the constant 72; the variant with 76 in its place is the ancillary file `R355_316_r3441e77.json` in `flag5_r34k/certificates/`, and the same checker accepts it without the input $R(3, 4, 4) \leq 73$. The manifest in that directory lists it with $R(3, 4, 5) \leq 137$ as its only input. The $(3, 4, 8)$ certificate uses 409 only through $R(3, 4, 7) \leq 411$, so that $D_3 = 410$.

The variants with starred inputs replace the published two-colour bounds by those of Table 3, and the one for $R(3, 4, 8)$ also uses $R(3, 4, 6) \leq 245$. $\square$

Where the weight sits. In the certificate for $R(3, 4, 4) \leq 73$, as in that of Theorem 8.2, the multiplier weight sits on rows with small caps. Most of it is on the cap rows

$$|N_i(x) \cap N_i(y)| \leq 8 \quad (xy \text{ of colour } i) \quad \text{and} \quad |N_1(x) \cap N_i(y)| \leq 8$$

for $i \in \{2, 3\}$, which come from $R(3, 4) = 9$. Most of the rest is on the three-rooted cap rows

$$|N_1(x) \cap N_i(y) \cap N_i(z)| \leq 3$$

for an edge yz of colour $i \in \{2, 3\}$, which come from $R(2, 2, 4) = 4$, and on the red degree cap 17. The degree cap 29 and the lower rows carry almost no weight, and no other cap row comes from $R(3, 3, 4) = 30$, so this value enters essentially only through the admissible profiles. The least slack in (ii) of Lemma 8.1 is attained, up to a small fraction of the margin, at the four admissible profiles with $d_1 = 17$ and at $(14, 29, 29)$. So the certificate rests on $R(3, 4) = 9$, through caps on common neighbourhoods, and it is tightest where the red degree reaches its cap 17 or the blue and green degrees both reach 29.

For comparison, $R(3, 4, 5) \leq 137$ and (1) give only $R(3, 5, 5) \leq 46 + 137 + 137 - 1 = 319$. Similarly, $R(3, 4, 4) \leq 73$ gives only $R(4, 4, 4) \leq 3 \cdot 73 - 1 = 218$. Theorem 8.3 supersedes Theorems 6.2, 6.3 and 6.4 and Proposition 6.5, whose hand proofs remain as short independent arguments for weaker bounds.

For $\mathbf{t} = (3, 3, k)$, Lemma 8.1 gives both exclusions and valid lower bounds $m_k(n)$, and the procedure of Section 5 can take the larger of these and its own at every order.

Theorem 8.4 (certificates checked by computer). $R(3, 3, 6) \leq 86$, $R(3, 3, 7) \leq 125$ and $R(3, 3, 8) \leq 174$.

Proof. Three certificates with $s = 5$, for $(k, n) = (6, 86)$, $(7, 125)$ and $(8, 174)$, checked in this order. They have 64, 37 and 71 rows, all cap and lower rows except one upper floor row in the third. They have 121, 144 and 106 squares, mostly at triangles. None has a profile row of the recursive kind. So they use only the inputs of Table 3 and, at $k = 7$ and 8, the bounds just proved, through the caps.

They are checked in exact arithmetic over the 193 orbits of $S_5 \times \mathbb{Z}_2$ on the 29,197 labelled colourings of K_5 with no red and no blue triangle, and over the admissible profiles. The largest value in (i) is 0 for each, and the least values in (ii) are about $1.9 \cdot 10^{-7}$, $3.1 \cdot 10^{-8}$ and $1.5 \cdot 10^{-8}$, which are therefore also the margins. $\square$

Theorem 8.5 lowers the third bound to 173. So Theorems 8.4 and 8.5 supersede Theorems 4.1, 4.3 and 4.5, whose hand proofs remain as short independent arguments for weaker bounds. The certificates of Theorem 8.3 use these weaker bounds as caps.

Theorem 8.5 (computer-assisted). $R(3, 3, k) \leq 173, 241, 309, 401, 505, 620, 748$ and 892 for $k = 8, \dots, 15$.

Verification. A single chain, the procedure of Section 5 with valid lower bounds also taken from certificates, is checked level by level by a program that does not import the generating code. At each level $k = 3, \dots, 15$ the program works as follows.

1. It first checks the certificates of that level. A recursive profile row is accepted only with values already accepted at level $k - 1$, or with the bound $\lceil z \max(0, z - 1 - C_{33})/2 \rceil$ at green degree z , as each vertex of $N_3(v)$ has at most C_{33} green neighbours in $N_3(v)$.
2. It then checks the rows of the ladder at level k . Each claimed $m_k(n)$ must be at most the largest of
 - $\lceil nh_{\min}/2 \rceil$,
 - the bounds from the recorded supporting lines of the functions F and Ψ of Lemmas 3.2 and 3.3,
 - Lemma 3.5(a) and (b), and
 - an accepted certificate value at (k, n) .
3. Finally, for $k \geq 6$, it checks the exclusion of B_k , by a certificate or by $\Psi > 0$ at every admissible h ; for $k \leq 5$ it checks that B_k is the input 17, 30 or 57.

No bound on $R(3, 3, k)$ with $k \geq 6$ is an input: each B_k is treated as infinite until the chain excludes it, so every bound B_j with $j \geq 6$ used at a level is proved earlier in the chain.

The chain has

- 116 *density certificates*, of the second form in Lemma 8.1, which give valid lower bounds $m_k(n)$, with $s = 4$ at levels 3 to 6,
- an exclusion of 87 with $s = 4$, and
- 38 certificates with $s = 5$: densities at levels 6, 7 and 8, the exclusions of Theorem 8.4, and a few that add nothing.

For $8 \leq k \leq 15$ the final exclusion is by Ψ . Without certificates, the same checker reproduces Table 4 exactly. $\square$

k	6	7	8	9	10	11	12	13	14	15
Sections 4 and 5	88	129	179	247	315	407	512	628	756	901
with certificates	86	125	173	241	309	401	505	620	748	892

Table 5: Upper bounds on $R(3, 3, k)$.

Remark 8.6 (Limits). At one order below each bound of Theorem 8.3, except the starred 245, below which 244 was not tried, the optimum of the semidefinite programme is 0 to solver precision, between $-8 \cdot 10^{-9}$ and $-6 \cdot 10^{-10}$. In particular we could not exclude $(3, 4, 4; 72)$. The optimal profile mixture puts weight 0.89 on $(17, 27, 27)$, and adding the exact pattern

counts of the unique $(4, 4; 17)$ -colouring [20] inside $N_1(v)$ when $d_1(v) = 17$ still gives no certificate.

For $R(3, 3, k)$ the five-vertex cutting-plane searches at $(3, 3, 6; 85)$, $(3, 3, 7; 124)$ and $(3, 3, 8; 172)$ were stopped without a certificate. When they were stopped, the cutting-plane relaxations still had points satisfying every cut with slack about $3 \cdot 10^{-7}$, $6.5 \cdot 10^{-8}$ and $2.5 \cdot 10^{-8}$; a certificate appears only once this slack becomes negative. So these searches are inconclusive: they found no certificate, but did not show that the optimum is 0. Neither the four- nor the five-vertex search excludes $(3, 3, 5; 56)$. With $s = 4$ nothing is gained for $R(3, 3, k)$ above $k = 6$.

The search of Theorem 8.3 gives no certificate at orders where colourings are known, for instance $(3, 3, 4; 29)$, $(3, 4, 4; 54)$ and $(4, 4, 4; 127)$.

8.3 Consequences for four colours

The bounds of Theorems 8.2 and 8.3 also give new bounds for four-colour numbers, which are defined as for three colours. In this subsection $\mathbf{t} = (t_1, \dots, t_4)$ and the $\mathbf{e}_i$ are the unit vectors.

The four-colour recurrence. The four-colour form of (1) [20, 6.1(a)] is

$$R(\mathbf{t}) \leq \sum_{i=1}^4 R(\mathbf{t} - \mathbf{e}_i) - 2,$$

and it improves to $R(\mathbf{t}) \leq \sum_i R(\mathbf{t} - \mathbf{e}_i) - 3$ when $\sum_i R(\mathbf{t} - \mathbf{e}_i)$ is even and some $R(\mathbf{t} - \mathbf{e}_i)$ is even.

The proof is that of (1) and of its parity refinement. In a colouring of K_n with no colour- j K_{t_j} for any j , the set $N_i(v)$ has no colour- i K_{t_i-1} and no colour- j K_{t_j} for $j \neq i$, so $d_i(v) \leq R(\mathbf{t} - \mathbf{e}_i) - 1$, and summing over i gives $n - 1 \leq \sum_i (R(\mathbf{t} - \mathbf{e}_i) - 1)$, that is, $n \leq \sum_i R(\mathbf{t} - \mathbf{e}_i) - 3$. At $n = \sum_i R(\mathbf{t} - \mathbf{e}_i) - 3$, which is odd under the parity hypothesis, every one of these caps is attained. So for an i with $R(\mathbf{t} - \mathbf{e}_i)$ even every vertex has odd colour- i degree, which contradicts the handshake lemma. As for three colours, proven upper bounds may replace the $R(\mathbf{t} - \mathbf{e}_i)$, and an entry equal to 2 may be dropped.

Corollary 8.7 (from computer-assisted inputs). $R(3, 3, 4, 4) \leq 442$, $R(3, 4, 4, 4) \leq 1521$ and $R(4, 4, 4, 4) \leq 6082$.

Proof. Apply the recurrence with $R(3, 4, 4) \leq 73$ (Theorem 8.3), $R(3, 3, 3, 4) \leq 149$ [20] and $R(4, 4, 4) \leq 198$ (Theorem 8.2):

$$\begin{aligned} R(3, 3, 4, 4) &\leq 2 \cdot 73 + 2 \cdot 149 - 2 = 442, \\ R(3, 4, 4, 4) &\leq 198 + 3 \cdot 442 - 3 = 1521, \\ R(4, 4, 4, 4) &\leq 4 \cdot 1521 - 2 = 6082. \end{aligned}$$

In the second line parity applies, as the sum $198 + 3 \cdot 442 = 1524$ and all four terms are even. In the first and third it does not, as every term is odd. $\square$

The inputs and earlier bounds. Theorems 8.2 and 8.3 are computer-assisted. The survey's $R(3, 3, 3, 4) \leq 149$ is itself the recurrence with parity, from $R(3, 3, 4) = 30$ [6] and $R(3, 3, 3, 3) \leq 62$ [8]: as $3 \cdot 30 + 62 = 152$ and 30 are even, it gives $152 - 3 = 149$.

The survey lists $R(3, 3, 4, 4) \leq 450$, $R(3, 4, 4, 4) \leq 1576$ and $R(4, 4, 4, 4) \leq 6306$, and notes that 6301 is implicit in [5]. The bound 6301 is $4 \cdot 1576 - 3$, the recurrence with parity, as $4 \cdot 1576$ and all four terms are even. With 450 in place of 442, that is, without $R(3, 4, 4) \leq 73$, the bound $R(4, 4, 4) \leq 198$ alone gives $R(3, 4, 4, 4) \leq 1545$ and $R(4, 4, 4, 4) \leq 6178$.

The survey's $R(3, 4, 4, 4) \leq 1576$, which it also marks as implicit in [5], comes from Boza's analogue modulo 3 of the parity refinement [5, Theorem 2.1], which counts the triangles of one colour. Stated for exact values, it lowers the right-hand side $P = \sum_i R(\mathbf{t} - \mathbf{e}_i) - 2$ of the recurrence by one when three conditions hold: $P \not\equiv 1 \pmod{3}$; for some i with $t_i \geq 4$, $R(\mathbf{t} - \mathbf{e}_i)$ equals its own right-hand side and is $\not\equiv 1 \pmod{3}$; and, for the same i , $R(\mathbf{t} - 2\mathbf{e}_i) \not\equiv 1 \pmod{3}$. From Boza's $R(4, 4, 4) \leq 229$ and the survey's $R(3, 3, 4, 4) \leq 450$ it gives $229 + 3 \cdot 450 - 2 - 1 = 1576$ (one may assume that these bounds, and $R(3, 4, 4) \leq 77$ and $R(3, 3, 3, 4) \leq 149$, are attained, as otherwise the recurrence already gives 1576). With our bounds it gives nothing more: at the three steps P is 442, 1522 and 6082 (before parity, which applies only at the second step), all $\equiv 1 \pmod{3}$, so the first condition fails.

Four triangle-free colours. We also tried to improve $R(3, 3, 3, 3) \leq 62$ [8], that is, to exclude colourings of K_{61} with four triangle-free colours, by certificates as in Sections 7 and 8, on four or five vertices and with four-colour analogues of the caps of Lemmas 2.1 and 2.3. In such a colouring each colour degree is at most $R(3, 3, 3) - 1 = 16$ [13] and the four sum to 60, so, up to the order of the colours, the profile of a vertex is one of

$$(12, 16, 16, 16), \quad (13, 15, 16, 16), \quad (14, 14, 16, 16), \quad (14, 15, 15, 16), \quad (15, 15, 15, 15).$$

Such a certificate is a dual solution of a flag-algebra relaxation [21], so it exists only if this relaxation is infeasible. The unknowns of the relaxation are the densities of the colourings of K_4 or K_5 in a putative colouring of K_{61} , subject to positive-semidefiniteness constraints and to linear constraints from the caps, such as $|N_i(v) \cap N_j(u)| \leq R(3, 3) - 1 = 5$ for $u \in N_i(v)$ and $j \neq i$, as in Lemma 2.1.

Both relaxations are numerically feasible: floating-point solvers found densities meeting all their constraints, even when every vertex is required to have profile $(12, 16, 16, 16)$. So a case split by vertex profiles would not help: the case in which every vertex has this profile is not excluded. The four-vertex relaxation was solved as a linear programme by HiGHS [15], with its semidefinite constraint imposed by eigenvector cuts until no eigenvalue was below -10^{-8} , and the five-vertex one as a semidefinite programme by Clarabel [12] at tolerance 10^{-8} . So we obtained no certificate, and no bound follows. For comparison, the six-vertex computation of Lidický and Pfender [19, Appendix A], which uses no caps, gives only $R(3, 3, 3, 3) \leq 65$.

9 Lower bounds from split circulants

The witnesses for $R(3, 3, k)$ with $k \leq 14$ in Table 2 are circulant colourings. The vertex set is $\mathbb{Z}_n$, and the *distance* of a pair $\{u, v\}$ is the $d \in \{1, \dots, \lfloor n/2 \rfloor\}$ with $u - v \equiv \pm d \pmod{n}$. Given disjoint sets $\Delta_1, \Delta_2 \subseteq \{1, \dots, \lfloor n/2 \rfloor\}$, colour $\{u, v\}$ red if its distance lies in Δ_1 , blue if it lies in Δ_2 , and green otherwise. Thus red is the Cayley graph of $\mathbb{Z}_n$ with connection set

$$\pm\Delta_1 := \{\pm a \bmod n : a \in \Delta_1\}.$$

As translations preserve colours, red has a triangle if and only if it has one through 0, say $\{0, x, x + y\}$. That is, it has one if and only if $x, y, x + y \in \pm\Delta_1$ for some x, y (the three vertices are then distinct, as $0 \notin \pm\Delta_1$). So red is triangle-free if and only if $\pm\Delta_1$ is *sum-free*: $x + y \notin \pm\Delta_1$ for all $x, y \in \pm\Delta_1$, including $x = y$. Likewise for blue and $\pm\Delta_2$. A green K_k is excluded separately, by clique search.

Colour splitting. The colourings were found by *colour splitting*, the method of Exoo and Tatarevic [7, §3.5]. It starts from a two-colour circulant on n vertices, the *seed*, whose second colour is K_k -free. A SAT solver, CaDiCaL 1.5 through PySAT [16], searches for a partition of the distances of its first colour into classes Δ_1 and Δ_2 with $\pm\Delta_1$ and $\pm\Delta_2$ sum-free. Green is then the seed's second colour, so the split is a $(3, 3, k; n)$ -colouring. As $R(3, 3) = 6$, a split can exist only if the first colour is K_6 -free.

The seeds. Our seeds are $(5, k)$ circulants of Kuznetsov, whose first colour is even K_5 -free, though a split still need not exist. For $k = 9, 11, 12, 14$ they are those of [17, Table 2]. For $k = 13$ the circulant there, on 232 vertices, admits no split, even without circulant symmetry: with one variable per edge the condition is unsatisfiable, already on any 51 consecutive vertices. We use instead a 231-vertex circulant $(5, 13)$ -colouring from the program distributed with the source files of [17].

We chose these seeds because their orders exceed those of the witnesses for the survey's lower bounds on $R(3, 3, k)$. For $k = 9, 11, 12, 14$ the new bound $R(3, 3, k) \geq n + 1$ equals the survey's lower bound on $R(5, k)$. For $k = 13$ it is one less than $R(5, 13) \geq 233$.

For $k = 10$ no known $(5, 10)$ -colouring is as large as the survey's 149-vertex witness for $R(3, 3, 10)$. The largest, of Exoo and Tatarevic [7] (see [17, Table 3]), has 148 vertices and is not a circulant. Although it splits edge by edge, it gives only $R(3, 3, 10) \geq 149$, one below the survey's $R(3, 3, 10) \geq 150$. For $k = 15$ we instead add vertices to another split of the seed for $k = 14$ (Section 9.1).

The splits are ours. The verification checks all three colours directly, so the validity of the witnesses does not depend on the seed or the solver.

Theorem 9.1 (witnesses checked by computer). *Let (n, k) be one of the pairs $(132, 9)$, $(182, 11)$, $(202, 12)$, $(231, 13)$ and $(266, 14)$. Colour the edges of the complete graph on $\mathbb{Z}_n$ as above, with Δ_1 and Δ_2 the distance sets listed below for this pair. This colouring has no red or blue triangle and no green K_k . Hence $R(3, 3, k) \geq n + 1$ (Table 2).*

Each item gives $|\Delta_1|$, $|\Delta_2|$ and, as a check on transcription, the degree profile (d_1, d_2, d_3) , which is common to all vertices. Each vertex has two neighbours at every distance $d < n/2$ and, when n is even, one at distance $n/2$, as $n/2 \equiv -n/2 \pmod{n}$. So a colour with distance set Δ has degree $2|\Delta|$, less 1 if n is even and $n/2 \in \Delta$. Also $d_3 = n - 1 - d_1 - d_2$.

- $(n, k) = (132, 9)$: $\Delta_1 = \{5, 9, 11, 12, 15, 19, 29, 32, 36, 39, 46, 53, 56, 59, 63, 66\}$, $\Delta_2 = \{6, 16, 17, 20, 27, 28, 30, 31, 38, 41, 42\}$; $|\Delta_1| = 16$, $|\Delta_2| = 11$, $n/2 = 66 \in \Delta_1$, profile $(31, 22, 78)$.
- $(n, k) = (182, 11)$: $\Delta_1 = \{29, 36, 43, 50, 54, 61, 63, 68, 70, 75, 77, 80, 81, 84, 87, 88, 91\}$, $\Delta_2 = \{4, 5, 7, 13, 21, 22, 30, 31, 39, 40, 48, 64, 65, 66, 74, 82, 83\}$; $|\Delta_1| = 17$, $|\Delta_2| = 17$, $n/2 = 91 \in \Delta_1$, profile $(33, 34, 114)$.

- $(n, k) = (202, 12)$: $\Delta_1 = \{2, 9, 15, 16, 22, 23, 28, 35, 41, 48, 49, 52, 60, 66, 73, 78, 79, 85, 86, 91, 92, 99\}$, $\Delta_2 = \{1, 7, 10, 19, 27, 33, 36, 45, 51, 59, 62, 71, 76, 88, 97, 101\}$; $|\Delta_1| = 22$, $|\Delta_2| = 16$, $n/2 = 101 \in \Delta_2$, profile $(44, 31, 126)$.
- $(n, k) = (231, 13)$: $\Delta_1 = \{3, 4, 5, 12, 13, 20, 21, 29, 31, 37, 46, 48, 54, 65, 76, 82, 93, 104, 110\}$, $\Delta_2 = \{2, 6, 10, 17, 18, 39, 42, 43, 47, 51, 55, 63, 67, 71, 75, 79, 83, 87, 91, 95, 99, 103\}$; $|\Delta_1| = 19$, $|\Delta_2| = 22$, n odd, profile $(38, 44, 148)$.
- $(n, k) = (266, 14)$: $\Delta_1 = \{6, 8, 9, 10, 11, 13, 39, 41, 42, 43, 44, 64, 65, 67, 68, 69, 89, 90, 92, 93, 94, 122, 123, 124, 125, 127\}$, $\Delta_2 = \{1, 46, 49, 52, 55, 58, 61, 66, 96, 99, 102, 105, 108, 111, 114, 117, 120, 126, 129, 131, 133\}$; $|\Delta_1| = 26$, $|\Delta_2| = 21$, $n/2 = 133 \in \Delta_2$, profile $(52, 41, 172)$.

Verification. The sum-free check. Red and blue are triangle-free by the sum-free criterion above, a finite check on $\pm\Delta_1$ and $\pm\Delta_2$.

Green cliques. As translations preserve colours, every green clique has a translate containing 0, so it suffices to find the largest green clique through 0. An exact branch-and-bound bitset search for cliques through vertex 0, `clique_check.cpp`, gives green clique numbers 8, 10, 11, 12 and 13 for $k = 9, 11, 12, 13$ and 14, that is, $k - 1$. So there is no green K_k .

An independent check. A second exact search, `check_colouring_full.cpp`, which shares no code with `clique_check.cpp`, works on the full adjacency matrices without using translation invariance. It confirms that there is no green K_k and no red or blue triangle. The bounds do not need the green K_{k-1} , but it shows that the colouring is not $(3, 3, k - 1)$ -avoiding, so it does not also give $R(3, 3, k - 1) \geq n + 1$. The colourings and programs, including the sum-free check, are in the ancillary files (Section 10). $\square$

9.1 Extending a split circulant

A witness for $k + 1$ can be built from one for k by adding vertices. Let H be a colouring of K_n with no red or blue triangle and with green clique number q , and let F be a colouring of K_{n+r} whose restriction to the first n vertices is H . We call the other r vertices *new*.

For a new vertex x let A_x be the set of vertices of H joined to x in red or blue. For a set S of new vertices let

$$A_S = \bigcup_{x \in S} A_x.$$

Every green clique of F is $S \cup T$, where S , its set of new vertices, is a green clique and T is a green clique of $H - A_S$. A vertex w of H *certifies* S if w and all its red and blue neighbours in H lie in A_S .

Lemma 9.2. *Let W be a set of certifiers of S that is a green clique of H . Then every green clique of $H - A_S$ has at most $q - |W|$ vertices.*

Proof. Let T be such a clique. It is disjoint from W , as $W \subseteq A_S$. A vertex of T is outside A_S , so it is not a red or blue neighbour of any $w \in W$, and is therefore a green neighbour of every $w \in W$. So $T \cup W$ is a green clique of H , and $|T| + |W| \leq q$. $\square$

Let W_S be the set of certifiers of S , and $\omega(W_S)$ the green clique number of H on W_S . By Lemma 9.2 with W a largest green clique in W_S , every green clique T of $H - A_S$ has

$|T| \leq q - \omega(W_S)$. So F has green clique number at most $q + 1$ if every green clique S of new vertices satisfies

$$|S| \leq 1 + \omega(W_S).$$

Checking this needs no clique search in F . It runs over the green cliques of new vertices, and needs only the green clique numbers of the sets W_S .

The simplest new vertex is a *copy*. The copy y_v of a vertex v of H has the colours of v towards $H - v$, and vy_v is green. So A_{y_v} is the set of red and blue neighbours of v . Then v certifies $\{x, y_v\}$ for every new vertex x with $v \in A_x$, and $\{y_u, y_v\}$ whenever uv is red or blue.

Theorem 9.3 (witness checked by computer). $R(3, 3, 15) \geq 278$.

Proof. Step 1: the colouring H . Let H be the circulant colouring of $\mathbb{Z}_{266}$ with $\Delta_1 = \{1, 42, 46, 49, 52, 55, 58, 61, 64, 96, 99, 102, 105, 108, 111, 114, 117, 120, 127, 129, 131, 133\}$ and $\Delta_2 = \{6, 8, 9, 10, 11, 13, 39, 41, 43, 44, 65, 66, 67, 68, 69, 89, 90, 92, 93, 94, 122, 123, 124, 125, 126\}$; $|\Delta_1| = 22$, $|\Delta_2| = 25$, $n/2 = 133 \in \Delta_1$, profile $(43, 50, 172)$. Both $\pm\Delta_1$ and $\pm\Delta_2$ are sum-free, and $\Delta_1 \cup \Delta_2$ is the same set as for the $(266, 14)$ witness of Theorem 9.1. So H is another split of the same seed, with the same green graph, and $q = 13$.

Step 2: the new vertices. Add eleven new vertices:

- the copies $y_0, y_5, y_{70}, y_{131}, y_{140}, y_{201}$ of the *anchors* $0, 5, 70, 131, 140, 201$; and
- five vertices $x_1, \dots, x_5$, each green to every vertex of H except as follows:
 - x_1 is red to 0 and 70 and blue to 201 ;
 - x_2 is red to 131 and 201 and blue to 70 ;
 - x_3 is red to 5 and blue to 140 ;
 - x_4 is red to 140 and blue to 201 ;
 - x_5 is red to 5 and blue to 70 .

Step 3: colours among the new vertices. Colour the pairs of new vertices, taken in the order $y_0, y_5, y_{70}, y_{131}, y_{140}, y_{201}, x_1, \dots, x_5$, by the following rows. Row i lists the colours from the i th new vertex to the eleven new vertices in this order (r red, b blue, g green; a dot marks the diagonal):

```
.rbggbgbrbrb r.bgrbbgrbg bb.rbrggbrg ggr.bbbgrbr grbb.rrbggb bbrbr.ggbgr
  bgbgrg.brrb brggbgb.rbr rbgbrgrr.bb rbrbggrbb.r bggrbrbrbr.
```

So $x_1, \dots, x_5$ span a red pentagon $x_1x_3x_2x_5x_4$ and a blue pentagram, and x_i is green to y_v exactly when x_iv is red or blue.

Step 4: no red or blue triangle. The resulting colouring F of K_{277} has no red or blue triangle, by a direct check.

Step 5: green cliques. The green edges among the new vertices are $y_0y_{131}, y_0y_{140}, y_5y_{131}$ and the twelve pairs x_iy_v with x_iv red or blue. They span no green triangle. Each has a certifier:

- 0 for the first two and 5 for the third, by the rule for copies, as the pairs $\{0, 131\}, \{0, 140\}$ and $\{5, 131\}$, at distances $131, 126$ and 126 , are red or blue in H ; and

- v for $x_i y_v$.

Let $S \cup T$ be a green clique of F , as before Lemma 9.2, so that $|S| \leq 2$. If $|S| \leq 1$, then $|S \cup T| \leq 1 + q$. If $|S| = 2$, then S has a certifier w , and Lemma 9.2 with $W = \{w\}$ gives $|T| \leq q - 1$. Either way $|S \cup T| \leq q + 1 = 14$. So F is a $(3, 3, 15; 277)$ -colouring. $\square$

Verification. The matrix and the certifiers. One program rebuilds the 277×277 matrix from Δ_1, Δ_2 , the anchors and the rows above, so its first 266 vertices carry H . Another, `certcheck.py`, reads the matrix, checks that red and blue have no triangle, and for every green clique S of new vertices computes the certifiers and checks the condition after Lemma 9.2. With $q = 13$ from Theorem 9.1, this is a complete check.

Independent searches. Independently, and without the lemma, `check_colouring_full.cpp` of Theorem 9.1 and a second exact branch-and-bound search on the full matrix find no red or blue triangle and no green K_{15} . The second search, and a third that computes $\max_S (|S| + \omega(H - A_S))$ exactly over the decomposition $S \cup T$, give green clique number 14. $\square$

The search and its limits. The anchors and the colours among the new vertices were found by a SAT search over copies of a set of anchors together with vertices that are green to H outside the anchors. In none of the 416 circulant splits of the seed, counted up to exchanging red and blue, does this family give more than eleven new vertices.

Call vertices u, v of H *compatible* if they have no common red neighbour or no common blue neighbour, as holds whenever uv is red or blue. Two copies y_u, y_v with uv green cannot be green to each other. Otherwise u, v, y_u and y_v would be pairwise green, and with a green K_{11} in the common green neighbourhood of u and v , whose vertices are also green to both copies, they would form a green K_{15} . Such a K_{11} exists for every green pair: a clique search over the pairs $\{0, d\}$ with d a green distance finds one, which suffices by translation invariance. Two copies y_u, y_v can be red only if u and v have no common red neighbour, and blue only if they have no common blue neighbour. So the anchors are pairwise compatible. In our enumeration of the splits, at most six vertices of H are pairwise compatible, so there are at most six anchors and six copies.

Two new vertices of the second kind cannot be green to each other either. Both are green to every vertex of H outside the at most six anchors, and H minus the anchors still contains a green K_{13} , which with the two of them would form a green K_{15} . Indeed, as H is vertex-transitive, a set meeting every green K_{13} has at least $266/13 > 20$ vertices. So every pair of them is red or blue, and as $R(3, 3) = 6$ there are at most five of them.

10 Verification, literature and limitations

Ancillary files. The ancillary files of this submission, also in the directory `ancillary/` of the repository <https://github.com/Lachy-Dauth/ramsey-neighbourhood-counting>, contain programs that check every computer-assisted result of the paper. Each certificate is checked by a program written from the definitions, which does not import the code that found it. The Python and C++ programs use only their standard libraries, except that the six-vertex checker in `flag5_r444/` and its negative controls use NumPy for integer counts. Statements

about searches and negative experiments (for instance in Sections 8 and 9, in Remark 8.6 and on $R(3, 3, 3, 3)$ in Section 8.3) report our runs and are not certified by these files.

The command `sh run_all.sh` runs the quick checks in a few minutes, `sh run_all.sh --full` adds the long checks, and `sh run_all.sh --controls` the negative and positive controls. The checks of Theorems 8.2, 8.3 and 8.5, the second checker for Theorem 8.2, the full-matrix searches of `check_colouring_full.cpp` and the direct clique searches for Theorem 9.3 run only with `--full`. Of Section 8, the quick mode checks Theorem 8.4 and the weaker bound $R(4, 4, 4) \leq 204$.

- `verify_hand_proofs.py`: the main steps of the proofs in Sections 4 and 6 (some intermediate values, such as gradients and the published-only reductions in the proof of Proposition 6.5, and some values quoted only in remarks, are not checked);
- `bootstrap.py`, which runs the procedure of Section 5 and writes its certificate `tables.json`, and `verify_bootstrap.py`, a checker that shares no code with it: Theorem 5.2;
- `verify_cascade.py` and `verify_cascade_published_only.py`, with and without the unpublished inputs: the program in the proof of Proposition 6.5;
- `verify_r444.py`: Lemma 7.3, Theorem 7.4, and identity (7) with the values of J in Theorem 7.5;
- `verify_r444_squares.py`: the four-vertex certificates with several squares for $n = 219, \dots, 224$;
- `flag5_r444/`: Theorem 8.2, with a second checker over six-vertex orbits;
- `flag5_r34k/`: Theorem 8.3, the variant of its $(3, 5, 5)$ certificate without $R(3, 4, 4) \leq 73$, and the mixture-model certificate for $R(4, 4, 4) \leq 200$, with a manifest that checks which bounds each certificate uses;
- `flag_r33k/`: Theorems 8.4 and 8.5, the whole chain of certificates and ladder rows;
- `circulants.json`, `check_circulants.py`, `clique_check.cpp` and `check_colouring_full.cpp`: Theorem 9.1; `check_circulants.py` regenerates the adjacency matrices of the five witnesses from Δ_1 and Δ_2 (the lists A and B of `circulants.json`) and prints the first 16 hexadecimal digits of their SHA-256 hashes;
- `extension_r3315/`: Theorem 9.3, with `certcheck.py` for the check through Lemma 9.2.

Corollary 8.7 has no program of its own: it follows by hand from Theorems 8.2 and 8.3, checked in `flag5_r444/` and `flag5_r34k/`, and from the survey's $R(3, 3, 3, 4) \leq 149$.

Tests and negative controls. All these checks were run, together with negative controls (perturbed certificates, constants changed by one, the order changed by one), all of which were rejected. For the flag certificates of Section 8, the rows were also evaluated on explicit colourings avoiding the target, where every row holds, and the square expansions were compared exactly with the squares on explicit colourings. These tests check the implementation. At the orders of the theorems no colouring exists, and the validity of the rows rests on their proofs in Section 8.1.

Formalisation in Lean. Two Lean 4 files in `lean/`, `LocalCombinatorics.lean` and `ConventionalDefinitions.lean`, which need Mathlib, formalise with no unproved steps (no sorry) two facts used in Lemma 3.1, on which all our hand bounds for $R(3, 3, k)$ rest.

- The first is the cap $|N_1(v) \cap N_2(u)| \leq k - 1$ for $u \in N_1(v)$ in a $(3, 3, k)$ -colouring, and its sum over $u \in N_1(v)$, whose left side is $2e_2(N_1(v))$. So $2e_2(N_1(v)) \leq (k - 1)d_1(v)$ (Lemma 2.1 and Corollary 2.2).
- The second is the pointwise identity behind $T(H) = \sum_v (e_2(N_1(v)) + e_1(N_2(v)))$ (Lemma 2.4). In a colouring with no red or blue triangle, each triangle of H contributes exactly 1 to the sum, at its apex, and every other triangle contributes 0.

Nothing else is formalised.

Literature. Comparison values are from revision 18 of the survey [20] and from [5]. For the four-colour numbers of Corollary 8.7 they are the bounds of [20] and [5] recalled after it and listed in Table 1. We searched arXiv and the web up to 23 October 2026 for earlier statements of our new bounds, and found none.

The search included the flag-algebra tables of Lidický and Pfender [19]. Among three-colour numbers with every $t_i \geq 3$, these bound only $R(3, 3, 4)$, $R(3, 3, 5)$, $R(3, 4, 4)$ and $R(3, 4, 5)$, by 32, 57, 85 and 406. None is below the bound known at the time. For $R(3, 3, 5)$ they match 57.

Such a search cannot establish priority, and we would welcome pointers to any earlier source.

Limitations. The procedure of Section 5 alone improves neither $R(3, 3, 5) \leq 57$ nor the hand bounds 88, 129 and 179 (Remark 5.1). With the certificates of Section 8 it gives Table 5. No method here improves $R(3, 3, 5) \leq 57$, on which every bound for $R(3, 3, k)$ with $k \geq 6$ depends.

One order below the bounds of Theorems 8.2 and 8.3, except the starred 245 (the order 244 was not tried), the five-vertex relaxations have optimum 0 to solver precision. We also found no five-vertex certificate for $(3, 3, 6; 85)$, $(3, 3, 7; 124)$ or $(3, 3, 8; 172)$ (Section 8 and Remark 8.6). Further progress for these numbers needs certificates on six or more vertices, new valid inequalities, or better inputs.

Corollary 8.7 rests on Theorems 8.2 and 8.3, which are computer-assisted, and on $R(3, 3, 3, 4) \leq 149$, which [20, 6.1(a)] derives from $R(3, 3, 4) = 30$ [6] and $R(3, 3, 3, 3) \leq 62$ [8], both computer-assisted. Our four- and five-vertex relaxations for $R(3, 3, 3, 3)$ are numerically feasible at $n = 61$ (Section 8.3), so they do not improve $R(3, 3, 3, 3) \leq 62$.

Searches in restricted families, namely circulant colourings and block and extension constructions, found no $(3, 3, 7; 85)$ -colouring and no $(3, 4, 4; 55)$ -colouring, which would improve the lower bounds $R(3, 3, 7) \geq 85$ and $R(3, 4, 4) \geq 55$ listed in the survey, respectively. A failure in a restricted family proves nothing in general.

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programs can be run independently of the AI systems. The author takes full responsibility for the contents of this paper.

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