Investigation on Computational Bounds of Ramsey Numbers and Properties of Extremal Ramsey Graphs

COMP3821 Research Project Report

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1 Introduction

1.1 Overview

Ramsey theory is a branch of combinatorics that studies the unavoidable patterns that appear in sufficiently large or complex structures. Some notable results include the Hales-Jewett theorem [3] on combinatorial lines in high-dimensional grids, the Erdős-Szekeres theorem [2] guaranteeing large convex polygons among planar point sets, and Van der Waerden's theorem [1] on the existence of monochromatic arithmetic progressions.

In this report we focus on Ramsey numbers, where $R(n_1, n_2, ..., n_c)$ is defined as the minimum number of vertices in a complete graph required so that any edge colouring with c colours contains at least one complete monochromatic subgraph of size n_i for colour i.

Historically, while there have been major developments in the broad field of Ramsey Theory, the rate of evaluating Ramsey numbers and their bounds has been relatively slow. This has changed over the past few decades, vastly to the credit of computer algorithms. Using large amounts of computation power has allowed researchers in the field to greatly improve both lower and upper bounds for many Ramsey Numbers, and for some cases evaluated some exact numbers [11].

1.2 Report Structure

We are interested in the computational methods and algorithms used to tighten the bounds on Ramsey numbers.

We have researched and summarised our findings on existing algorithmic techniques for 2 and 3 colour Ramsey numbers, particularly exploring *gluing algorithms* in depth, and attempting to extend this 2 colour technique to 3 or more colours.

In an effort to make this technique more efficient by reducing the number of cases that must be considered, we propose and attempt to prove 2 conjectures, additionally writing new algorithms assuming the conjectures are true.

We additionally explored *cyclic graph generation*, a technique that has been used to increase lower bounds for Ramsey graphs.

2 Existing 2-colour Proof Methods

2.1 Overview

The standard problem for computing Ramsey Numbers of the form $R(k, l)$, requires finding the smallest integer n such that every graph with n vertices contains a clique of order k, or an independent set of order l. Note in this section, we are not presenting the problem as a colouring problem searching for monochromatic cliques of size k or l. Instead, we define the problem of finding cliques of size k and independent sets of size l. This is in line with how the problem is represented in other papers on the topic.

Classical proof techniques that are heavily reliant on manual calculation have not been effective at proving the exact value of many Ramsey Numbers. In fact, there only exist known computer-free proofs for the exact values of 7 Ramsey-Numbers [9]. These numbers are recorded in A.1.1.

Refer to Appendix A.1.2 for a table containing some known values and bounds. This table was sourced from Radziszowski's survey on Ramsey Numbers [11].

More modern approaches use complex algorithms that require huge amounts of computation to improve existing bounds. Even improving a bound by one is considered an impressive result in the field. For example, in 2024 Vigleik Angeltveit and Brendan Mckay were able to improve the upper bound of $R(5, 5)$ from 48 to 46. This improvement took around seven years, as the two had reduced the upper bound to 48 in 2017 [14].

Not all Ramsey proofs focus on finding specific numerical bounds: in 1935, Erdős used a probabilistic approach to prove that the upper bound for a Ramsey number $R(k, k) \leq 4^k$[2]. Only in 2024 was this bound able to be tightened, as researchers Campos, Griffiths, Morris, and Sahasrabudhe were able to show that $(4 - \epsilon)^k$ was in fact a sufficient upper bound [19], and gave proofs for unoptimised values of ϵ, namely $2^{-10}, 2^{-7}$.

We are interested in methods of proof that heavily utilise computer assistance, particularly those involving *gluing algorithms*.

2.2 Gluing

At a high level, these proofs begin by showing that Ramsey graphs of a given size can be split into two disjoint subgraphs, each of which is also a Ramsey graph, and consequently enumerate these two families and algorithmically *glue* them together to show that the original Ramsey graph cannot exist. There are a variety of methods and additional steps used in order to reduce the size of the search space, ensure completeness and verify results. By nature of being computationally expensive it is primarily used where bounds are already tight: with lower order Ramsey graphs.

2.2.1 Definitions and Algorithmic Foundations

To reiterate, we will temporarily reframe the problem from a *two-coloured complete graph with coloured cliques of order* a, b to an *uncoloured incomplete graph with a clique of order* a *or an independent set of order* b.

Let $RG(s, t)$ be a simple graph with no clique of size s and no independent set of size t, and $\mathcal{R}(s, t)$ be the set of all (s, t) graphs. Define $RG(s, t; n)$ as an $RG(s, t)$ of order n, and $\mathcal{R}(s, t; n)$ similarly.

For a graph $G = (V, E)$, we denote G_v^+, the *neighbourhood of* v, as the induced subgraph of all vertices adjacent to v in G, and similarly G_v^-, the *dual neighbourhood of* v, as the induced subgraph of all vertices not v or not adjacent to v in G. It is clear that if G is an $RG(s, t; n)$ and $v \in V$ has degree d, then G_v^+ is an $RG(s-1, t; d)$ and G_v^- is an $RG(s, t-1; n-1-d)$.

2.2.2 Method used in $R(4, 5) = 25$[6]

The upper bound $R(4, 5) \leq 25$ was established by McKay and Radziszowski (1995) by an exhaustive computer-assisted search that enumerated all possible gluings of suitable $RG(3, 5)$ and $RG(4, 4)$ neighbourhoods and into a class of $RG(4, 5; 24)$, defined such that any $RG(4, 5; 25)$ must be a 1 vertex extension of one of these graphs. They found that no valid extension existed, and thus that $R(4, 5)24$, tightening the existing bound enough to state that $R(4, 5) = 25$. The search used the facts that $R(3, 5) = 14$ and $R(4, 4) = 18$ to restrict the possible degree d of a vertex in a hypothetical $RG(4, 5; 25)$ graph to

$$7 \leq d \leq 13,$$

and proceeded by gluing representatives from sets of $RG(3, 5; d)$ and $RG(4, 4; 24-d)$ graphs, completely enumerating the set of $RG(4, 5; 24)$. A naive gluing approach was found to be insufficiently efficient, such that several optimisations were necessary.

To clarify requirements, the gluing algorithm is required to combine representatives from $RG(3, 5; d)$, $RG(4, 4; 24-d)$ into a single $RG(4, 5; 24)$, by adding edges between the vertices of the two representatives. Let these two subgraphs be $G = (V_G, E_G), H = (V_H, E_H)$ respectively, and let the combined graph be $F = (V, E)$. Now define a *feasible cone* $C_v \subseteq V_H$ for some $v \in V_G$ as the vertices in H which are adjacent to v, or equivalently $F_v^+ \cap V_H$. It is clear that C_v must not cover a clique of order 3, otherwise F would contain a clique of order 4. Our gluing problem is now to choose feasible cones for every $v \in G$, such that no undesired independent sets or cliques appear in F.

Again, a naive brute force of these feasible cones would be too inefficient, thus the cones were grouped into structured families which could be analysed in parallel. These *intervals* are a set of feasible cones denoted as $[B, T] = \{X | B \subseteq X \subseteq T\}$, for some feasible cones B, T. A series of endofunctions were defined to simplify a given cone, and used within some *collapsing rules* that apply to sequences of intervals. These collapsing rules are able to continually and dramatically reduce the number of potential gluing operations, even being able to prune entire families – the authors attribute the success of the method to these operations.

2.2.3 Method used in $R(5, 5) \leq 46$[20]

To prove this upper bound (and the previous $R(5, 5) \leq 46$ in [14]), Angeltveit and McKay used a method they coined as *gluing along an edge*. Given a hypothetical $(5, 5, 46)$ graph G, this method considers two adjacent vertices a, b, their neighbourhoods G_a^+, G_b^+ and the 'overlap' K of these neighbourhoods, such that $K = (G_a^+)_b^+ = (G_b^+)_a^+$. As both G_a^+, G_b^+ are $RG(4, 5)$ graphs, a enumeration of valid pairs of these $RG(4, 5)$ was compiled, by finding pairs within $\mathcal{R}(4, 5)$ which had an isomorphic overlap K. Valid pairs were attempted to be glued together, to enumerating all possibilities to prove that no $RG(5, 5; 46)$ existed.

Compiling these valid pairs also required excessive computation given the sheer number of possibilities. To limit this, the authors additionally used linear programming methods to prune the required elements of $\mathcal{R}(4, 5)$ graphs limiting them to order $21, 22, 23, 24$ and with a carefully chosen number of edges for each; varying edge counts affected the strength of the proof and affected subsequent components. These sets were generated using more gluing operations of smaller graphs.

2.3 Cyclic Ramsey Graph Generation

An alternative computational approach for finding lower bounds on Ramsey numbers involves the systematic generation of cyclic (or circulant) Ramsey graphs. Developed by Kuznetsov [12], this method constructs large Ramsey graphs with cyclic symmetry—graphs invariant under rotation of vertex labels. Many of the largest known Ramsey graphs exhibit this property.

The method uses *distance colorings*, which generalize cyclic colorings: a distance coloring of a complete N-graph satisfies $C(a, b) = C(a+k, b+k)$ for any valid indices. For smaller Ramsey numbers, full enumeration is feasible using depth-first search with optimizations including forced link search (identifying near-complete cliques to force edge colors), forced link group search, out-of-order coloring, and periodic graph rebuilds. This approach successfully enumerated distance colorings for pairs up to $R(6, 7)$.

For larger instances where full enumeration is intractable, Kuznetsov developed a connected component search starting from extensible signatures. New signatures are discovered through nearest-neighbor search (bit flipping) and vertex relabeling operations. Initial seed signatures are found using operations on cyclic colorings: bit flip, reflection, and cyclic vertex permutation. Starting from small Ramsey colorings and iteratively applying these operations while increasing the minimum acceptable size explores the space of large cyclic graphs without exhaustive enumeration.

This implementation successfully found colorings for many Ramsey number pairs, and was particularly effective for cases like $R(6, 8)$ and $R(6, 9)$ where other approaches become intractable, making it a promising direction for improving computational bounds.

3 Existing 3-colour Proof Methods

3.1 Overview

A natural extension to the classical Ramsey Numbers is to consider edge colourings using more than two colours. The classical general Ramsey Number is of the form $R(G_1, ..., G_r)$, where each G_i is the monochromatic subgraph corresponding to colour i. The most commonly considered multicolour Ramsey Numbers however are the diagonal cases where each of the G_i's are complete graphs of the same size. These numbers are denoted by $R_r(m)$, where r is the number of colours, and m is the size of the monochromatic cliques [7].

The added complexity of extending these Ramsey Numbers to more than two colours has made it extremely difficult to calculate exact values of these numbers. In fact, as of 2024, there are only two multicolour Ramsey numbers whose exact values are known. These are $R_3(3) = 17$ and $R(K_3, K_3, K_4) = 30$. Furthermore, even the general behaviour of these numbers is relatively unknown. In fact, Erdos and Sos conjectured in 1979 [4] that

$$\lim_{m \mapsto \infty} \frac{r(K_3, K_3, K_m)}{r(K_3, K_m)} = \infty.$$

The consequence of this is that being able to predict the behaviour of $r(K_3, K_m)$ does not necessarily provide us with useful information about the multicolour case $r(K_3, K_3, K_m)$.

3.1.1 Existing Lower Bounds for Multicolour Ramsey Numbers

In his comprehensive survey on Ramsey Numbers, Radziszowski includes the tables in A.1.3 detailing the known lower bounds of the small diagonal multicolour Ramsey Numbers [11]. It is clear to see that the sudden increase in size of these multicolour Ramsey Numbers is even more extreme than that of the standard 2-colour numbers.

From our survey of existing proofs and methods for the computation of 3 colour Ramsey numbers, as well as more general n colour Ramsey numbers, we came to the realisation of the significant increase in difficulty of computing Ramsey Numbers as the number of colours are increased. We therefore shifted our focus towards algorithms for 2-colour Ramsey Numbers which could potentially then be extended for the more general case.

3.2 Calculating bounds

Whilst exact values for even 3 colour Ramsey numbers are extremely difficult to calculate, there are many interesting theorems about the bounds on 3 colour Ramsey numbers for special types of graphs. For example, it has been proven that for every even n, $r(C_n, C_n, C_n) = 2n + o(n)$ as $n \to \infty$ [10]. In fact, in the same paper, the authors prove that there exists n_1 such that for every even $n > n_1$, $r(C_n, C_n, C_n) = 2n$. To prove these results, the paper extends arguments that were proven about the simpler $r(P_n, P_n, P_n)$ Ramsey numbers. Note that a similar result has been proven for $r(C_n, C_n, C_n)$ where n is odd [8].

Recently, there have been significant improvements to the bounds of many special Ramsey numbers. These bounds were achieved by using the flag algebra method. Specific improvements and an in-depth walkthrough of the method can be found in [16]. This method involves finding a certificate for the bound as a sum of squares. This can be done efficiently using semidefinite programming. Flag algebras are generally used to solve problems with asymptotic results. Since Ramsey problems, specifically Ramsey numbers, are finite, researchers have considered the blow-up of a Ramsey graph to transfer the problem to an asymptotic setting [18].

The work of Rowley has been instrumental in establishing improved bounds for several multi-colour Ramsey numbers. He utilised SAT solvers to help generate the colourings for ”template” graphs which are graphs holding some desirable Ramsey properties. He then combined these template graphs in the construction of larger compound graphs which inherit the Ramsey properties. From these compound graphs Rowley was able to tighten many lower bounds [17].

3.3 3-Colour Gluing

This section serves to explore the possibility of extending 2-colour gluing methods towards 3 colours.

First, we must explore the possible subgraph selection that can be made. We can use a ’coloured neighbourhood’ method, similar to what has been done for 2 colours, or perhaps attempting to glue two 2-colour Ramsey graphs together.

Extending our earlier definitions in the 2-colour gluing section, define an $RG(a, b, c)$ as a simple complete graph with 3-coloured edges without a red-coloured K_a, blue-coloured K_b and green-coloured K_c. Define $RG(a, b, c; n)$ as an $RG(a, b, c)$ with n vertices, and G^r_v for some tri-coloured graph G as the induced subgraph of all vertices connected to v via a red edge, and similarly for G^b_v, G^g_v. Define the number of red edges connected to some v as d^r_v, and similarly for d^b_v, d^g_v.

Now let us consider the first case: for a given $RG(a, b, c; n) = (V, E)$, it is clear that, for any vertex v V, G^r_v must be an $RG(a - 1, b, c; n - 1 - d^r_v)$, and similarly for the other two colours. This leaves us with 3 subgraphs, $RG(a-1, b, c; n-1-d^r_v)$, $RG(a, b-1, c; n-1-d^b_v)$, $RG(a, b, c-1; n-1-d^g_v)$ and the vertex v. To rejoin them, we must consider all edge colourings between all 3 subgraphs with do not result in an undesired monochromatic clique. However, this results in $3^{n_1 \cdot n_2}$ edge colouring options between subgraphs of size n_1, and n_2. This, combined with the fact there are a large number valid graphs that may be considered means that, even with some excessive optimisations, any algorithm may be too inefficient.

As opposed to attempting to glue 3 subgraphs together directly, it may be more optimal to search for 2 possible subgraphs.

4 Extension Conjecture

The first algorithm we began to develop involved constructing Extremal Ramsey Graphs by building on smaller known Extremal Ramsey Graphs. The idea for this algorithm originated from a property that we noticed in some small Ramsey Graphs, and so we conjectured it to be true. We called this conjecture the Extension Conjecture.

Definition 1. *An extremal Ramsey graph, ERG, is a 2-colour graph on $R(a,b)-1$ vertices with the maximal number of edges such that there does not exist a monochromatic K_a or K_b subgraph in ERG. We denote the extremal Ramsey graph with $R(a,b)-1$ vertices as*

$$ERG(a,b).$$

Conjecture 1 (Extension). *Any extremal Ramsey graph $ERG(a,b)$ contains an $ERG(a,b-1)$ subgraph.*

A sketch of our attempted proof for the Extension Conjecture can be found in Appendix A.2. However, after multiple unsuccessful attempts to complete the proof we decided to computationally check whether or not it holds for some small Extremal Ramsey Graphs.

4.1 Disproof

Despite the intuitive appeal of the conjecture, we were able to disprove it through computational analysis of known extremal Ramsey graphs.

To test this conjecture, we obtained datasets of extremal Ramsey graphs from the ANU Mathematical Sciences Institute database [13]. Specifically, we analysed:

- The unique extremal $R(3,5)$ graph on 13 vertices (stored in `subgraphs.txt`)
- All 352366 extremal $R(4,5)$ graphs on 24 vertices (stored in `supergraphs.txt`)

The `conjecture-1.py` script implements a systematic verification approach. For each extremal $R(4,5)$ graph, we check all possible induced subgraphs on 13 vertices to determine if any are isomorphic to the known extremal $R(3,5)$ graph. The algorithm uses NetworkX's subgraph isomorphism functionality to perform this check efficiently. The fact that there is only one extremal $R(3,5)$ graph [5] makes this verification particularly efficient.

The key insight is that if the Extension Conjecture were true, then every one of the 352366 extremal $R(4,5)$ graphs would contain the unique extremal $R(3,5)$ graph as an induced subgraph. However, our computational analysis revealed multiple counterexamples—extremal $R(4,5)$ graphs that do not contain the extremal $R(3,5)$ subgraph. We have presented one such graph, alongside the $ERG(3,5)$ in Appendix A.3.

This disproof has important implications for algorithm design. We cannot rely on a recursive decomposition strategy that assumes extremal Ramsey graphs contain smaller extremal Ramsey graphs as building blocks. The structure of extremal Ramsey graphs appears to be more complex than this conjecture suggested, requiring more sophisticated approaches for their generation and analysis.

The complete implementation and data files are available in our GitHub repository at `github.com/Lachy-Dauth/3821-Ramsey`, with the main verification script being `conjecture-1.py`.

5 Degree Difference Conjecture

The second algorithm we developed relied on the conjectured property that vertices in Extremal Ramsey Graphs have very similar degrees. We formalised this conjecture as the Degree Difference Conjecture.

Conjecture 2 (Degree Difference). *No pair of vertices in extremal Ramsey graphs have degrees that differ by more than 1.*

If true, this property would significantly constrain the structure of extremal Ramsey graphs and potentially enable more efficient generation algorithms. A graph satisfying this property is said to be *regular* (if all degrees are equal) or *almost regular* (if degrees differ by at most 1).

However, we found this claim to be false. Regardless, we created a theoretical algorithm for graph generation under the assumption that the conjecture is true.

5.1 Disproof

Before attempting to prove the conjecture, we decided to perform a computational validation of our conjecture against McKay's collection of extremal Ramsey graphs [13] as we did for the Extension Conjecture. We wrote a simple Python script to read in the graphs and ensure that the maximum and minimum vertex degree differed by at most one. This script, `conjecture-2.py`, is accessible in the same repository as our other code.

The conjecture held for all $ERG(3,4)$, $ERG(3,5)$, $ERG(3,9)$, and $ERG(4,4)$ graphs tested. However, it failed for 3 of the 191 $ERG(3,7)$ graphs on 22 vertices. The existence of these graphs serves as counterexamples for the conjecture, and hence our conjecture does not hold in general. A visualisation of one of these counterexample graphs can be found in Appendix A.4.

Moreover, despite the intuitive appeal that extremal Ramsey graphs should have somewhat evenly distributed degrees, further investigation found this definitively not to be the case. Among the 350904 $ERG(4,5)$ graphs, we discovered 1208 examples containing vertices with degrees of 6 and 12, a difference of 6. This substantial degree variation far exceeds the modest irregularities we initially anticipated and demonstrates that extremal graphs can exhibit extreme structural asymmetry.

These counterexamples demonstrate that extremal Ramsey graphs can have significantly non-uniform degree sequences. Furthermore, this irregularity suggests that extremal graphs may exploit asymmetric structures to avoid monochromatic cliques, rather than relying on the symmetric, near-regular configurations that the Degree Difference Conjecture proposed.

The failure of this conjecture indicates that generation algorithms cannot assume near-regularity as a constraint. Instead, algorithms must explore a broader search space that includes highly irregular degree sequences. This increases the computational complexity of finding extremal Ramsey graphs but provides a more accurate understanding of their possible structures.

All code and data for this analysis are available in the repository at `github.com/Lachy-Dauth/3821-Ramsey`, with the main verification script being `conjecture-2.py`.

6 Algorithms

6.1 Naive

As part of our research, we have looked into existing methods for computing extremal Ramsey graphs for 2 Colour Ramsey Graphs. Firstly, consider the following statements.

- Let an extremal graph R have k vertices. Then, as there is an edge between any 2 vertices u, v, there is exactly $\binom{k}{2}$ edges in R.
- To check whether there is a clique of size n within any graph, it takes $O\left(\binom{k}{n} \cdot n^2\right)$ time in the naive brute-force approach, since we must examine all $\binom{k}{n}$ subsets of n vertices and verify whether each forms a complete subgraph.

Now, we can construct the following non-optimal algorithm. Given a target Ramsey number $R(a, b)$:

1. Initialize $k = 1$.
2. While true:
   (a) Generate all possible colourings of the complete graph K_k. There are $2^{\binom{k}{2}}$ such colourings.
   (b) For each colouring:
      i. Check whether there exists a red clique of size a or a blue clique of size b.
      ii. If no such monochromatic clique exists, record the colouring as a valid colouring.
   (c) If no valid colouring exists for current k, then return $R(a, b) = k$.
   (d) Otherwise, increment k by 1.

This algorithm will take the following amount of time to complete

$$T = \sum_{k=1}^{R(a,b)} 2^{\binom{k}{2}} \left(\binom{k}{a} a^2 + \binom{k}{b} b^2 \right).$$

where $2^{\binom{k}{2}}$ is the number of possible colourings with k vertices and $\binom{k}{a} a^2 + \binom{k}{b} b^2$ is the time needed to check for a cliques and b cliques in the graph. This sum is dominated by the final term, so asymptotically, so time complexity is

$$T = \Theta\left(2^{\binom{R(a,b)}{2}} \left(\binom{R(a,b)}{a} a^2 + \binom{R(a,b)}{b} b^2 \right) \right).$$

6.2 Using Extension Conjecture

Let $L_k^{(a,b)}$ denote a set of non-isomorphic colourings with k vertices and no red K_a or blue K_b subgraph. Then, we can construct the following algorithm, where given a target Ramsey number $R(a,b)$:

1. Initialize $k = 0, a^* = 1, b^* = 1$.
2. While $L_k^{(a^*,b^*)} \neq \varnothing$:
   (a) If $k = 0$ then initialize the library of valid colourings on no vertices
   $$L_0^{(a^*,b^*)} = \{\varnothing\}.$$
   (b) Otherwise, construct $L_k^{(a^*,b^*)}$ by extending every graph in $L_{k-1}^{(a^*,b^*)}$ as follows:
      i. Initialize $L_k^{(a^*,b^*)} = \varnothing$.
      ii. For each colouring $G \in L_{k-1}^{(a,b)}$:
         A. Create a new vertex v_k.
         B. Create new assignments of colours to the $k-1$ incident edges $\{(v_k, u) \mid u \in V(G)\}$.
         C. For every completed assignment yielding a full colouring C on k vertices with no red K_a and no blue K_b, insert it into $L_k^{(a^*,b^*)}$ if it is not isomorphic to an existing element.
3. If $L_k^{(a^*,b^*)} = \varnothing$ then
   (a) If $a = a^*$ and $b = b^*$ return $R(a,b) = k$.
   (b) Else if $a^* < a$, increment a^*, and continue to step 4, then 2, instead extending every maximal graph in $L_{k-1}^{(a^*-1,b^*)}$ to $L_k^{(a^*,b^*)}$.
   (c) Else if $b^* < b$, increment b^*, and continue to step 4, then 2, instead extending every maximal graph in $L_{k-1}^{(a^*,b^*-1)}$ to $L_k^{(a^*,b^*)}$.
4. Increment k by 1.

If the conjecture is true, it is easy to see that the algorithm is correct as an extremal (a,b) graph E always contains an extremal $(a,b-1)$ graph and all of the subgraphs in E satisfy no red K_a and blue K_b. Thus, the algorithm will always produce this graph through a sequence of vertex additions, which provides a minimal bound for $R(a,b)$. However, as this graph is by definition extremal, it is also a maximal bound, meaning the algorithm produces the correct $R(a,b)$ value.

Now, let $E_t^{(a,b)} = |L_t^{(a,b)}|$ be the number of non-isomorphic graphs on t vertices that can be extended. Then, the time complexity of this algorithm can be calculated as

$$T = \sum_{k=1}^{R(a,b)} E_{k-1}^{(a,b-1)} \cdot 2^{k-1} \left(\binom{k-1}{a-1} a^2 + \binom{k-1}{b-1} b^2 \right),$$

where 2^{k-1} is the upper bound of edge assignments for the new vertex, and the clique-checks are restricted to cliques containing the newly added vertex. This sum is dominated by its final term, so asymptotically

$$T = \Theta\left(E_{R(a,b)-1}^{(a,b-1)} \cdot 2^{R(a,b)-1} \left(\binom{R(a,b)-1}{a-1} a^2 + \binom{R(a,b)-1}{b-1} b^2 \right) \right),$$

significantly improving on the naive bound.

6.3 Using Degree Difference Conjecture

Similarly to the previous algorithm, let $L_k^{(a,b)}$ denote a canonical set of non-isomorphic colourings on k vertices with no red K_a and no blue K_b and $E_t^{(a,b)} = |L_t^{(a,b)}|$. Then we can create the following algorithm.

1. Initialize $k = 1$.
2. While $L_k^{(a,b)} \neq \varnothing$:
   (a) If $k = 0$ then initialize the library of valid colourings on no vertices
   $$L_0^{(a,b)} = \{\varnothing\}.$$
   (b) Otherwise, construct $L_k^{(a,b)}$ by extending every graph in $L_{k-1}^{(a,b)}$ as follows:
      i. Initialize $L_k^{(a,b)} = \varnothing$.
      ii. For each colouring $G \in L_{k-1}^{(a,b)}$:
         A. Create a new vertex v_k.
         B. Create new assignments of colours to the $k-1$ incident edges $\{(v_k, u) \mid u \in V(G)\}$.
         C. For every completed assignment yielding a full colouring C on k vertices with no red K_a and no blue K_b, with all vertices having a degree difference of at most 1 insert it into $L_k^{(a,b)}$ if it is not isomorphic to an existing element.
3. If $L_k^{(a,b)} = \varnothing$ then return $R(a, b) = k$.
4. Increment k by 1.

Once again, if the conjecture is true it is easy to see that the algorithm is correct as only this subset of graphs is checked to be extremal.
Let $B_{\deg}(k-1)$ denote the maximum number of allowed incident-edge assignments for the new vertex on a graph of size $k-1$ under the degree-difference constraint. The time complexity is therefore

$$T \;=\; \sum_{k=1}^{R(a,b)} E_{k-1}^{(a,b)} \cdot B_{\deg}(k-1) \left(\binom{k-1}{a-1} a^2 + \binom{k-1}{b-1} b^2 \right),$$

where $B_{\deg}(k-1)$ replaces the 2^{k-1} factor from unconstrained extension.
This sum is dominated by its final term, so asymptotically

$$T \;=\; \Theta\!\left(E_{R(a,b)-1}^{(a,b)} \cdot B_{\deg}(R(a,b)-1) \left(\binom{R(a,b)-1}{a-1} a^2 + \binom{R(a,b)-1}{b-1} b^2 \right) \right).$$

Since $B_{\deg}(k-1)$ is significantly smaller than 2^{k-1}, this yields a substantial reduction compared with unconstrained enumeration, although not as big as the Extension Conjecture.

6.4 Cyclic 3-Colour Generation

We have adapted Kuznetsov's cyclic graph generation algorithm [12] to the 3-colour domain, specifically targeting lower bounds for $R(3, 5, 5)$. This algorithm searches for cyclic graphs that avoid monochromatic cliques of specified sizes.

6.4.1 Representation

The algorithm operates on cyclic graphs where the vertices are integers $0, 1, \ldots, N-1$. The colour of an edge between vertex i and vertex j is determined solely by the circular distance $d = \min(|i-j|, N-|i-j|)$. Thus, a graph of size N is fully defined by a distance vector $D = [c_1, c_2, \ldots, c_{\lfloor N/2 \rfloor}]$, where $c_i \in \{0, 1, 2\}$ represents the colour of edges with distance i. This representation reduces the search space from $3^{N(N-1)/2}$ to $3^{\lfloor N/2 \rfloor}$.

6.4.2 Algorithm Steps

The algorithm employs an evolutionary search strategy:

1. **Initialization**: A pool of small, valid cyclic colourings is generated randomly.
2. **Relabeling (Isomorphism Search)**: For each candidate, we apply transformations of the form $x \to (M \cdot x) \mod N$, where $\gcd(M, N) = 1$. This generates isomorphic forms of the graph, potentially unlocking new growth paths that were blocked in the original distance configuration.
3. **Reflection (Growth)**: To increase the lower bound, we construct a candidate of size $N+1$ from a graph of size N. This is achieved by "reflecting" the distance vector: combining the prefix and suffix of the current distance sequence to form a longer sequence.
4. **Mutation**: To escape local optima, we apply two types of mutations:
   - *Bit Flip*: Changing the colour of a specific distance.
   - *Bit Swap*: Swapping the colours of two different distances. This operation was added specifically for the 3-colour case to better navigate the complex constraints of multicolour Ramsey graphs.
5. **Selection**: Candidates are checked against the forbidden clique sizes (e.g., no Red K_3, Blue K_5, Green K_5). Only valid graphs are retained for the next generation.

This approach allows us to efficiently explore the space of cyclic graphs and has the potential to find new lower bounds for multicolour Ramsey numbers where exhaustive search is infeasible. This was rather fruitful in the case of 2-colour Ramsey numbers.

6.4.3 Results

Ramsey Number	Known Lower Bound	Our Cyclic Bound
$R(3,3,4)$	30	24
$R(3,3,5)$	45	39
$R(3,4,4)$	55	50
$R(3,5,5)$	139	113

Table 1: Comparison of our cyclic lower bounds with known values for $R(3,k,m)$.

Our cyclic generator produced lower bounds that are consistently within 80% of the best known values. These results demonstrate that cyclic graphs can provide strong lower bounds, although no new bounds could be produced or old bounds replicated. Our implementation can be found in the GitHub repository at `github.com/Lachy-Dauth/3821-Ramsey`, specifically in the `ramsey_3colour_generator.py` file and the graph6 output files in the `results.md` directory.

7 Conclusion

7.1 Summary of Findings

The correctness of both of the algorithms we designed were reliant on the Extension and Degree Difference Conjectures. Whilst attempting to prove the Extension Conjecture, we encountered difficulties in completing the proof. We decided to computationally explore the conjecture, and found a counterexample to disprove the conjecture. Before attempting to prove the Degree Difference Conjecture, we also decided to test it on some known graphs, where once again we found a counterexample.

Although we ultimately disproved both conjectures on which our algorithms were based, applying them to some small instances allowed us to correctly compute the corresponding Ramsey numbers. Since the conjectures do not hold in general, the algorithms cannot be guaranteed to succeed in all cases. Nevertheless, it was encouraging that they produced accurate results for some small Ramsey Numbers.

Through our implementation of cyclic Ramsey graph generation, we were able to obtain some promising lower bounds for several Ramsey numbers. Whilst the bounds we were able to find were not as tight as the best known bounds for those numbers, we were able to show that the approach does have merit, and that potentially with the right improvements and optimisations it could be used to find new or improve existing bounds.

7.2 Future work

In our initial surveying we noticed that 3-colour Ramsey numbers, have not been closely studied, with values known only for $R(3, 3, 3)$ and $R(3, 3, 4)$. It maybe possible to extend and apply existing neighbourhood gluing techniques, although it would require the enumeration of these large graph classes.

While we have found counterexamples to disprove the Extension and Degree Difference Conjectures in general, we leave open the question of whether they might still hold for specific types of graphs.

Our work on cyclic Ramsey graph generation represents a direction with significant merit that warrants further optimisation. This outcome raises the possibility that the strong cyclic symmetry exploited by this method may be a feature more prevalent or advantageous in 2-colour Ramsey graphs, making it less effective for the multicolor case. However, our implementation was a preliminary exploration. We believe that a fully optimised implementation, incorporating all of Kuznetsov's advanced techniques (such as forced link search and connected component exploration) and running on high-performance compute with larger time budgets, could prove otherwise. With more computational resources and a refined implementation, this method may yield new lower bounds for multicolour Ramsey numbers.

Lastly, our algorithms both focused on improving upper bounds of Ramsey Numbers and were specifically focused on utilising an approach similar to previously implemented gluing algorithms. In our survey of the field, we found very interesting approaches to improving lower bounds of Ramsey Numbers. These approaches included using SAT solvers and flag algebra to construct instances of Ramsey Graphs which tightened previously known lower bounds. We pose the question of whether some of these methods could be adapted to improve upper bounds, and whether a gluing style approach could be used for lower bounds as well.

A Appendix

A.1 Currently Known Values and Bounds

A.1.1 Proven in a Computer-Free Method

- $R(3, 3) = 6$
- $R(3, 4) = 9$
- $R(3, 5) = 14$
- $R(3, 6) = 18$
- $R(3, 7) = 23$
- $R(4, 4) = 18$

A.1.2 2-Colour bounds

Table 2: Values and bounds for $R(k, \ell)$, $k \leq 10$, $\ell \leq 15$ [11]

$k \backslash \ell$	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40-41	47-50	53-59	60-68	67-77	74-87
4		18	25	36-41	49-61	59-84	73-115	92-149	102-191	128-238	138-238	147-349	158-417
5			43–46	59-87	80-143	101-216	133-316	149-442	183-633	203-848	233-1138	267-1461	275-1878
6				102–165	115–298	134–495	183–780	204–1171	262-1804	294–2566	347-3703	?-5033	401-6911
7					205–540	219–1031	252–1713	292–2826	405–4553	417–6954	511–10578	?-15263	?-22112
8						282–1870	329–3583	343–6090	457–10630	?–16944	817–27485	?-41525	873-63609
9							565–6588	581–12677	?–22325	?–38832	?-64864		
10								798-23556	?-45881	?-81123			1313-?

A.1.3 3-Colour bounds

Table 3: Known lower bounds for diagonal multicolor Ramsey numbers $R_r(m)$ [15]

r\m	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
3	17	128	454	1106	3214	7174	15041	23094
4	51	634	4073	23502	94874	182002	719204	
5	162	4176	41626	258506				
6	538	32006	441606					
7	1698	160024						
8	5288							
9	17805							

Radziszowski also includes some lower bounds for non-diagonal Ramsey Numbers of three colours of the form $R(3, k, m)$. Note that $R(3, k, m)$ corresponds to the Ramsey graph containing no K_3 of colour 1, K_k of colour 2 and K_m of colour 3.

Table 4: Known lower bounds for $R(3, k, m)$ [15]

k\m	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
3	30	45	61	85	103	129	150	174	194	217	242	269	291
4	55	89	117	152	193	242							
5	89	139	181	241									

A.2 Proof Attempt for Extension Conjecture

Lemma A.1. *For every a, b, there exists an extremal Ramsey Graph $ERG(a, b)$ that is isomorphic to the complete graph $K_{R(a,b)-1}$.*

Proof. The proof follows directly from the definition of a Ramsey Number. $R(a, b)$ is by definition the smallest number k such that a 2-colour complete graph with k vertices must contain either a monochromatic K_a or K_b subgraph.

Now suppose that the complete graph on $R(a, b)-1$ vertices always contains either a monochromatic K_a or K_b. This would mean that $R(a, b) = R(a, b) - 1$ which is a contradiction.

Therefore there must exist a 2-colouring of $K_{R(a,b)-1}$ such that there is no monochromatic K_a or K_b in the graph. Which by definition is the extremal Ramsey graph $ERG(a, b)$. □

Lemma A.2. $R(a, b) - R(a, b - 1) \geq a \qquad \forall a \geq 3.$

Proof. This follows from a result in S.A. Burr, P. Erdo¨s, R.J. Faudree and R.H. Schelp, On the Difference between Consecutive Ramsey Numbers, Utilitas Mathematica, 35 (1989) 115-118. Proof to be adapted and included later. □

Theorem A.3. *Every extremal Ramsey graph $ERG(a, b)$ has an extremal Ramsey graph $ERG(a, b-1)$ as a subgraph.*

Proof. Suppose that there exists an extremal Ramsey graph on $R(a, b) - 1$ vertices that does not have an $ERG(a, b - 1)$ subgraph. We will call this graph G. This means that for every set of $R(a, b - 1) - 1$ vertices in G we either have a monochromatic K_a or a monochromatic K_{b-1} sub-subgraph.

But we know that no subgraph of G can have a monochromatic K_a sub-subgraph since by definition, G does not have a monochromatic clique of size a.

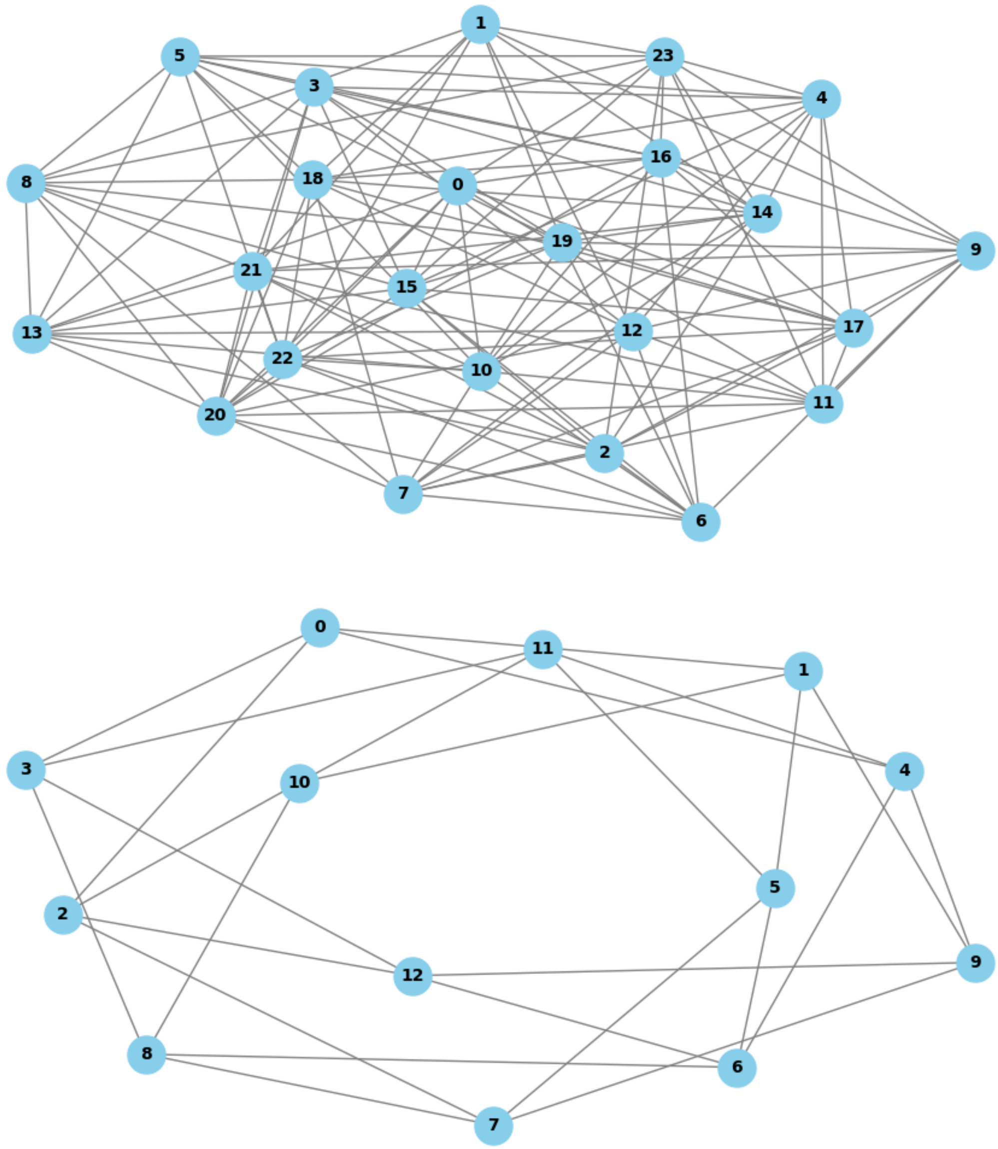
This means that every subset of $R(a, b - 1) - 1$ vertices of G must contain a monochromatic clique of size $b - 1$.

It remains to show that this implies a K_b clique in G, and thus cannot be true.

□

Note that we were unable to complete the proof from this point.

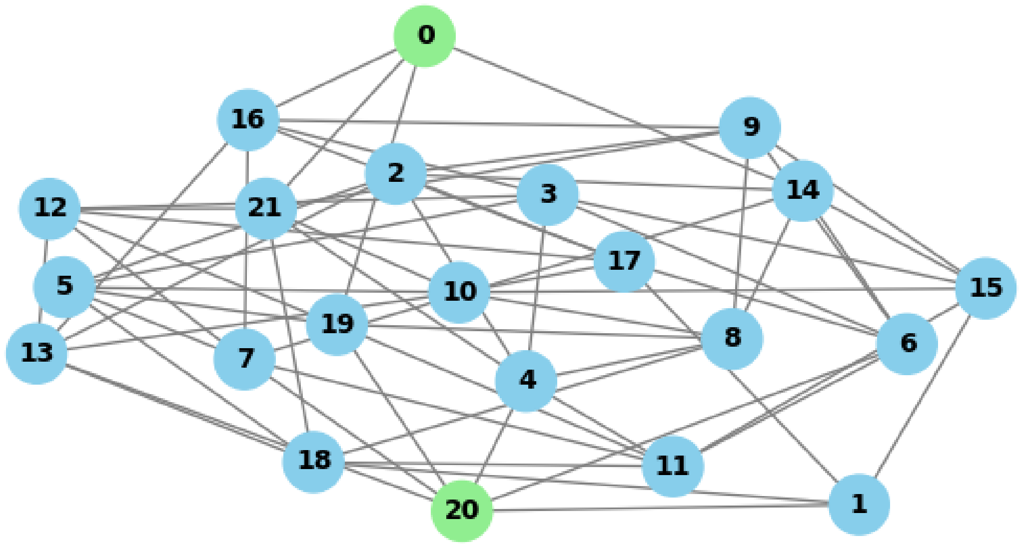
A.3 Extension Conjecture Counterexample



Only red edges are shown for clarity.

Figure 1: Counter-example: $ERG(5,4)$ (top) that does not contain the only $ERG(5,3)$ (bottom)

A.4 Degree Difference Conjecture Counterexample



Only red edges are shown for clarity.

Figure 2: Counter-example of Degree Difference Conjecture: an $ERG(3, 7)$ with vertices 0 and 20 differing by two degrees.

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